Letters in Nonlinear Analysis and its Applications **2** (2024) No. 3, 105-120
Available online at www.lettersinnonlinearanalysis.com
Research Article

Letters in Nonlinear Analysis and its Applications
Peer Review Scientific Journal
ISSN: 2958-874x

On some properties of the *alpha*-order and the weight distribution functions

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Abstract

In this paper, we describe how it can be applied the α-order to any fuzzy sets in a great class $\mathcal{C}$ of fuzzy sets. Before that, we introduce the algebraic tools that are underlying to this construction, and we detail the mentioned class of fuzzy sets of the real line that can be ranked by using such fuzzy binary relation. Furthermore, we introduce some new properties both of the weight distribution functions and of the α-orders. For the sake of completeness, we illustrate all processes by showing examples enough.

Keywords: fuzzy set, fuzzy number, alpha-ordering, aggregation function, binary relation.
2010 MSC: 03E72, 47S40, 46A55

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Received: February 2, 2024, Accepted: August 25, 2024, Online: September 12, 2024

1. Introduction

Real numbers and their operations are the basis of Mathematics. With them, we can represent the natural phenomena that occur around us. However, real numbers are often not enough to describe certain ambiguities that naturally appear when studying some processes.

Fuzzy numbers are a probabilistic way of extending real numbers to situations of uncertainty. Since Zadeh's fuzzy sets appeared in [1] as tools to represent indeterminacy, the idea was to expand the notion of real number to more general environments, using fuzzy tools and interpretations (see [2] for historical details on fuzzy sets). This led to the introduction of the concept of *fuzzy number* and its corresponding algebra (see [3, 4, 5]). However, today, the definition of fuzzy number takes on different nuances from one author to another (see [6, 7, 8, 9, 10]), and it is not globally accepted.

One of the main difficulties of fuzzy numbers is their ordering. Although real numbers are canonically ordered on the real line, there is no universally accepted methodology for ordering fuzzy numbers that verifies all the desired properties (see [11, 12, 13]). In this work we adopt the view-point proposed in [14]. However, other procedures stand out for their interesting proposals (see, for instance, [13]).

In a very recent paper, Neres *et al.* [15] introduced the notion of α-*order* (also called *alpha-order* for bibliographic purposes). This is a binary relation defined on a class $\mathcal{C}$ of fuzzy sets of the real numbers containing the family of all fuzzy numbers on $\mathbb{R}$. Its definition involves some algebraic tools that, at first sight, are not trivial. However, we do believe that the α-order has a great future because it satisfies several properties that are according to human intuition (for instance, it is admissible, see [16, Theorem 4.1]).

Since the α-order may seem somewhat complicated to be computed, in this paper we show how to apply the α-order in a very easy way. To do that, we describe and study the first and the second weight distribution functions. We illustrate how to successively apply them, in such a way that we can compute all the necessary vectors to work out the α-order. Taking advantage of this study, we explain a simple way to carry out α-orderings by employing fuzzy sets that are more general than fuzzy numbers. We illustrate all the procedures by showing examples enough. Finally, we present some new properties of the weight distribution functions and the α-order.

2. Preliminaries

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N} = \{1, 2, \ldots\}$ the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ and let denote by $\mathbb{I}$ to the closed and bounded interval $[0, 1]$. Given $n \in \mathbb{N}$, we denote $J_n = \{1, 2, \ldots, n\}$ and $J_n^0 = J_n \cup \{0\} = \{0, 1, 2, \ldots, n\}$.

Henceforth, let X be a non-empty set. Let X^n stands for the Cartesian product $X \times X \times \overset{(n)}{\cdots} \times X$. Given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we denote by $\langle \mathbf{u}, \mathbf{v} \rangle$ to the *scalar product* of both vectors, that is, if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{j=1}^n u_j v_j$.

2.1. Binary relations

A *binary relation* on X is a non-empty subset $\mathcal{R} \subseteq X \times X$. For simplicity, if $(x, y) \in \mathcal{R}$, we denote it by $x \preceq y$. We will say that x and y are $\preceq$-*related* (or $\preceq$-*comparable*) if $x \preceq y$ or $y \preceq x$ (or both). A binary relation $\preceq$ on X is: *reflexive* if $x \preceq x$ for all $x \in X$; *antisymmetric* if we can deduce $x = y$ when $x, y \in X$ satisfy $x \precsim y$ and $y \precsim x$; *transitive* if we can deduce $x \preceq z$ for each $x, y, z \in X$ satisfying $x \preceq y$ and $y \preceq z$; *total* (or *complete*) if $x \preceq y$ or $y \preceq x$ for all $x, y \in X$ (each two points of X are $\preceq$-related); a *preorder* if it is reflexive and transitive; an *order* (or a *partial order*) if it is reflexive, antisymmetric and transitive; a *preference relation* if it is transitive and total; a *total order* (or a *linear order*) if it is total and an order. A complete study of binary relations can be found in [17].

2.2. Fuzzy sets and fuzzy numbers of the real line

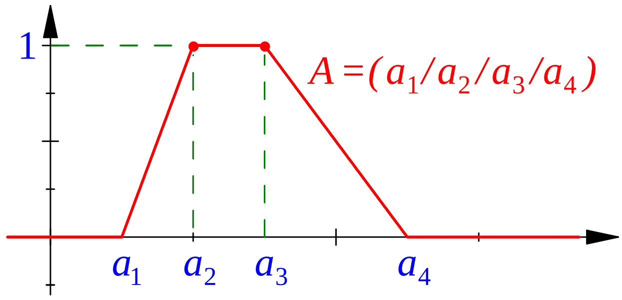
For our purposes, a *fuzzy set A on X* is a function $A : X \to \mathbb{I}$. The set X is the *universe* (or *universe of discourse*) in which the fuzzy set is considered. For each $x \in X$, the value $A(x)$ is the *membership degree* in which the point x belongs to the fuzzy set A. A fuzzy set $A : X \to \mathbb{I}$ is *normal* if there is $x_0 \in X$ such that $A(x_0) = 1$. For each $\beta \in (0, 1]$, the β-*level set* (or β-*cut*) of A is the set $A_\beta = \{ x \in X : A(x) \geq \beta \}$. The *kernel* (or *core*) *of* A is $\ker(A) = A_1$, and the *support of* A is the set $\operatorname{supp}(A) = \{ x \in X : A(x) > 0 \}$. If $\beta, \beta' \in (0, 1]$ and $\beta \leq \beta'$, then $\ker(A) = A_1 \subseteq A_{\beta'} \subseteq A_\beta \subseteq \operatorname{supp}(A) \subseteq X$. When X is endowed with a topology, it is usual to denote by A_0 to the closure of the support of A in the considered topology (which is a closed set of the topology). Notice that the set A_0 can only be considered when X is endowed with a topology and, in such a case, for each $\beta, \beta' \in (0, 1]$ such that $\beta \leq \beta'$,

$$\ker(A) = A_1 \subseteq A_{\beta'} \subseteq A_\beta \subseteq \operatorname{supp}(A) \subseteq A_0 \subseteq X.$$

Fuzzy numbers of the real line forms a particular class of fuzzy sets whose universe of discourse is $\mathbb{R}$. In order not to enter in a great discussion, we avoid the definition of *fuzzy number* (it can be seen on [13, 14]). Anyway, among them, we highlight some families of fuzzy numbers. For instance, a *trapezoidal fuzzy number* is a fuzzy set $A : \mathbb{R} \to \mathbb{I}$ of the real line for which there are four real numbers $a_1, a_2, a_3, a_4 \in \mathbb{R}$, with $a_1 \leq a_2 \leq a_3 \leq a_4$, such that the membership function A is defined as in Figure 1.

$$A(t) = \begin{cases} \dfrac{t - a_1}{a_2 - a_1}, & \text{if } a_1 < t < a_2, \\ 1, & \text{if } a_2 \leq t \leq a_3, \\ \dfrac{a_4 - t}{a_4 - a_3}, & \text{if } a_3 < t < a_4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Membership function.



(b) Trapezoidal fuzzy number.

Figure 1: Membership function and graphic representation of a trapezoidal fuzzy number.

The numbers a_1, a_2, a_3 and a_4 are called the *corners* of the trapezoidal fuzzy number, and the trapezoidal fuzzy number is denoted by $A = (a_1/a_2/a_3/a_4)$. We will stand $\text{TraFN}(\mathbb{R})$ for the family of all trapezoidal fuzzy numbers of $\mathbb{R}$. When $a_1 < a_2 < a_3 < a_4$, the membership function of $(a_1/a_2/a_3/a_4)$ is continuous, and its graphic representation shows the shape of a trapezoid whose mayor basis is the segment $[a_1, a_4]$ and whose minor basis is the segment $[a_2, a_3]$. Notice that when $a_1 = a_2$ or $a_3 = a_4$, the membership function of the fuzzy number $(a_1/a_2/a_3/a_4)$ is not continuous at $t = a_2$ or at $t = a_3$, respectively. If $r \in \mathbb{R}$ and $a_1 = a_2 = a_3 = a_4 = r$, the fuzzy number $(r/r/r/r)$ is denoted by $\widetilde{r}$, it is known as *crisp fuzzy number* and its membership function $\widetilde{r} : \mathbb{R} \to \mathbb{I}$ is defined by $\widetilde{r}(t) = 1$, if $t = r$, and $\widetilde{r}(t) = 0$, otherwise. The family of all crisp fuzzy numbers generalizes, to the fuzzy setting, the set of all real numbers. If $a_2 = a_3$, the trapezoidal fuzzy number is called *triangular*. Triangular fuzzy numbers are written as $(a_1/a_2/a_3)$. We denote by $\text{TFN}(\mathbb{R})$ the family of all triangular fuzzy numbers of the real line. Using the previous notation, $\mathbb{R} \equiv \{ \widetilde{r} : r \in \mathbb{R} \} \subset \text{TFN}(\mathbb{R}) \subset \text{TraFN}(\mathbb{R})$.

2.3. Aggregation functions of the real line

Originally, an *aggregation function* was defined as a function $\mathcal{A} : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ satisfying the following properties: (1) *monotonicity*: $\mathcal{A}$ is increasing, that is, if $(t_1, t_2), (s_1, s_2) \in \mathbb{I} \times \mathbb{I}$ are such that $t_1 \leq s_1$ and $t_2 \leq s_2$, then $\mathcal{A}(t_1, t_2) \leq \mathcal{A}(s_1, s_2)$; (2) *boundary conditions*: $\mathcal{A}(0, 0) = 0$ and $\mathcal{A}(1, 1) = 1$. This notion attracted the attention of many researchers. In a few years, many extensions of the concept of aggregation function was introduced (for instance, to several input variables). A comprehensive study about aggregation functions can be found in [18].

When the interval $\mathbb{I}$ is excessively reduced, the natural extension of this notion to $\mathbb{R}$ necessarily loss the boundary constraint. In this work, an *extended aggregation function of the real line* will be a function $\mathcal{A}$:

$\bigcup_{m\in\mathbb{N}}\mathbb{R}^m \to \mathbb{R}$ satisfying the monotonicity condition, that is, for each $m \in \mathbb{N}$ and each $(t_1, t_2, \ldots, t_m), (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$ such that $t_j \leq s_j$ for all $j \in J_m$, we can deduce that $\mathcal{A}|_{\mathbb{R}^m}(t_1, t_2, \ldots, t_m) \leq \mathcal{A}|_{\mathbb{R}^m}(s_1, s_2, \ldots, s_m)$. When $m = 1$, we agree that $\mathcal{A}|_{\mathbb{R}}$ is the identity map on $\mathbb{R}$. For simplicity, we call *aggregation function* to an extended aggregation function of the real line, and we denote $\mathcal{A}|_{\mathbb{R}^m}(t_1, t_2, \ldots, t_m)$ simply by $\mathcal{A}(t_1, t_2, \ldots, t_m)$ (see also [16]).

2.4. The weight distribution functions

From now on, let $n \in \mathbb{N}$ be a natural number (which will be constant throughout this work). A *membership (degree) vector* is a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0,1]^n$ (with positive components) and a *weight vector* is a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{I}^n$ such that $\omega_1 + \omega_2 + \ldots + \omega_n = 1$ (its components can be zero, but they must sum 1). We denote by $\Omega_n = (0,1]^n$ to the family of all membership degree vectors and by W_n to the family of all weight vectors. Given a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in W_n$, let denote:

$$\min_+(\omega) = \min(\{\omega_1, \omega_2, ..., \omega_n\} \backslash \{0\}).$$

For simplicity, given $r \in \mathbb{R}$, we denote $\mathbf{r}_n = (r, r, \ldots, r) \in \mathbb{R}^n$. It will be of importance the vectors $\mathbf{0}_n = (0, 0, \ldots, 0) \in \mathbb{R}^n$ and $\mathbf{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

In order to define the first and the second weight distribution functions, the following mappings must be considered.

- $\eta : W_n \to \mathcal{P}(J_n)$ defined as $\eta(\omega) = \{\, j \in J_n : \omega_j = \min_+(\omega) \,\}$;
- $\overline{\eta} : W_n \to \mathcal{P}(J_n)$ defined as $\overline{\eta}(\omega) = \{\, j \in J_n : j \notin \eta(\omega)$ and $\omega_j > 0 \,\}$;
- $s : W_n \to \mathbb{R}$ defined as $s(\omega) = \sum_{j\in\eta(\omega)} \omega_j$.

For each $\alpha \in \Omega_n$ we define the following mappings:

- $\Lambda_\alpha : W_n \to \mathcal{P}(\mathbb{I})$ defined as $\Lambda_\alpha(\omega) = \{\, \alpha_j : j \in \eta(\omega) \,\}$;
- $\theta_\alpha : W_n \to \mathcal{P}(J_n)$ defined as $\theta_\alpha(\omega) = \{\, j \in J_n : \alpha_j = \min(\Lambda_\alpha(\omega)) \,\}$;
- $\overline{\theta}_\alpha : W_n \to \mathcal{P}(J_n)$ defined as $\overline{\theta}_\alpha(\omega) = \{\, j \in J_n : j \notin \theta_\alpha(\omega)$ and $\omega_j > 0 \,\}$;
- $s_{\theta_\alpha} : W_n \to \mathbb{R}$ defined as $s_{\theta_\alpha}(\omega) = \sum_{j\in\theta_\alpha(\omega)} \omega_j$.

Using this notation, we are able to consider the two weight distribution functions:

$$\vartheta, \varphi_\alpha : W_n \cup \{\mathbf{0}_n\} \to W_n \cup \{\mathbf{0}_n\}$$

as follows. The *first weight distribution* is the mapping $\vartheta : W_n \cup \{\mathbf{0}_n\} \to W_n \cup \{\mathbf{0}_n\}$ given by:

$$\vartheta(\omega)(j) = \begin{cases} 0, & \text{if } \omega_j = 0 \text{ or } j \in \eta(\omega), \\ \omega_j + \dfrac{s(\omega)}{\operatorname{card}(\overline{\eta}(\omega))}, & \text{otherwise.} \end{cases}$$

The *second weight distribution* is the mapping $\varphi_\alpha : W_n \cup \{\mathbf{0}_n\} \to W_n \cup \{\mathbf{0}_n\}$ given by:

$$\varphi_\alpha(\omega)(j) = \begin{cases} 0, & \text{if } \omega_j = 0 \text{ or } j \in \theta_\alpha(\omega), \\ \omega_j + \dfrac{s_{\theta_\alpha}(\omega)}{\operatorname{card}(\overline{\theta}_\alpha(\omega))}, & \text{otherwise.} \end{cases}$$

Notice that $\vartheta(\omega)$ only depends on ω, but $\varphi_\alpha(\omega)$ directly depends both on α and on ω. We denote by ϑ^0 to the identity on $W_n \cup \{\mathbf{0}_n\}$, by ϑ^1 to ϑ and by ϑ^k to the composition $\vartheta \circ \vartheta^{k-1}$ for all $k \geq 2$. Similarly, φ_α^0 will be the identity on $W_n \cup \{\mathbf{0}_n\}$, φ_α^1 will be φ_α and φ_α^k will be the composition $\varphi_\alpha \circ \varphi_\alpha^{k-1}$ for all $k \geq 2$. It was proved on [15, Proposition 7] that $\vartheta^k(\omega) = \varphi_\alpha^k(\omega) = \mathbf{0}_n$ for all $k \geq n$, so the vectors $\vartheta^k(\omega)$ and $\varphi_\alpha^k(\omega)$ are only interesting for $k < n$.

One of the main aims of this paper is to explain how the mappings ϑ and φ_α act in practice (see Section 3).

2.5. The α-order

The *α-order* (also called *alpha-order*) is a binary relation on a subset $\mathcal{C}$ of fuzzy sets of $\mathbb{R}$ satisfying some reasonable properties. It was very recently introduced by Neres *et al.* in [15]. As we shall comment, it is a preference relation (that is, a total preorder), but, actually, it is not an order because it is not antisymmetric. Let us recall its definition.

Let denote by $\mathcal{C}$ to the class of all fuzzy sets $A : \mathbb{R} \to \mathbb{I}$ satisfying the following properties: (1) normality, (2) bounded support, (3) for each $\beta \in (0, 1]$, the β-cut A_β is a closed subset of $\mathbb{R}$ with a finite number of connected components. The class $\mathcal{C}$ contains the set of all fuzzy numbers of $\mathbb{R}$. However, it is wider because it includes fuzzy sets of $\mathbb{R}$ that are not fuzzy numbers (see [15]).

Given $A \in \mathcal{C}$ and $\beta \in (0, 1]$, the β-level set A_β can be expressed as the union of a finite set of closed and bounded real intervals, that is,

$$A_\beta = I_1^{A,\beta} \cup I_2^{A,\beta} \cup \ldots \cup I_{r(A,\beta)}^{A,\beta},$$

where each $I_j^{A,\beta}$ is a closed and bounded real interval and $r(A, \beta) \in \mathbb{N}$ is the number of connected components of A_β. Notice that each two distinct intervals are incompatible, that is, $I_{j_1}^{A,\beta} \cap I_{j_2}^{A,\beta} = \varnothing$ for each $j_1, j_2 \in J_{r(A,\beta)}$ such that $j_1 \neq j_2$. Such intervals are determined by their extremes in such a way that $I_j^{A,\beta} = [\, a_j^{A,\beta}, b_j^{A,\beta}\,]$ for all $j \in J_{r(A,\beta)}$. The interval $I_j^{A,\beta}$ is a *singleton* if $a_j^{A,\beta} = b_j^{A,\beta}$, that is, if it contains a unique point. Anyway, let denote by:

$$\mathbb{I}_\beta^A = \left\{ a_1^{A,\beta}, b_1^{A,\beta}, a_2^{A,\beta}, b_2^{A,\beta}, \ldots, a_{r(A,\beta)}^{A,\beta}, b_{r(A,\beta)}^{A,\beta} \right\}$$

to the union of all extremes of connected components of A_β. We denote this set as

$$\mathbb{I}_\beta^A = \left\{ c_1^{A,\beta}, c_2^{A,\beta}, \ldots, c_{s(A,\beta)}^{A,\beta} \right\}$$

when repetitions of real numbers (which come from singletons) are removed and its elements are ordered as $c_1^{A,\beta} < c_2^{A,\beta} < \ldots < c_{s(A,\beta)}^{A,\beta}$. The number of elements in $\mathbb{I}_\beta^A$ (avoiding repetitions) is $s(A, \beta) \in \mathbb{N}$, which verifies $s(A, \beta) \leq 2r(A, \beta)$.

Given an (extended) aggregation function $\mathcal{A} : \bigcup_{m\in\mathbb{N}} \mathbb{R}^m \to \mathbb{R}$ and $\beta \in (0, 1]$, we denote by $\mathcal{A}_\beta^A$ to the aggregated value $\mathcal{A}(c_1^{A,\beta}, c_2^{A,\beta}, \ldots, c_{s(A,\beta)}^{A,\beta})$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega_n$ is a membership vector, let denote by $\mathcal{A}_\alpha^A$ to the n-dimensional vector $(\mathcal{A}_{\alpha_1}^A, \mathcal{A}_{\alpha_2}^A, \ldots, \mathcal{A}_{\alpha_n}^A) \in \mathbb{R}^n$.

Given a membership vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega_n$, a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in W_n$ and an aggregation function $\mathcal{A} : \bigcup_{m\in\mathbb{N}} \mathbb{R}^m \to \mathbb{R}$, associated to each fuzzy set $A \in \mathcal{C}$, let consider the real numbers:

$$v_{\alpha,\omega,\vartheta,\mathcal{A}}^k(A) = \langle\, \vartheta^k(\omega), \mathcal{A}_\alpha^A \,\rangle \qquad \text{and} \qquad v_{\alpha,\omega,\varphi_\alpha,\mathcal{A}}^k(A) = \langle\, \varphi_\alpha^k(\omega), \mathcal{A}_\alpha^A \,\rangle, \tag{1}$$

where $k \in \mathbb{N}_0$, $\langle \cdot, \cdot \rangle$ is the scalar product and ϑ and φ_α are the weight distribution functions. These numbers are called the *k-valuations of A relative to ϑ* (or to φ_α). For simplicity, when the membership vector α, the weight vector ω and the aggregation function $\mathcal{A}$ are given by the context, we denote $v_{\alpha,\omega,\vartheta,\mathcal{A}}^k(A)$ by $v_\vartheta^k(A)$ and $v_{\alpha,\omega,\varphi_\alpha,\mathcal{A}}^k(A)$ by $v_{\varphi_\alpha}^k(A)$.

Using the previous algebraic tools, we introduce the binary relation $\overset{\alpha}{\leq}$ on $\mathcal{C}$ as follows. Given $A, B \in \mathcal{C}$, we will write $A \overset{\alpha}{\leq} B$ if one of the following conditions holds:

(a) either $v_\vartheta^k(A) = v_\vartheta^k(B)$ and $v_{\varphi_\alpha}^k(A) = v_{\varphi_\alpha}^k(B)$ for all $k \in J_{n-1}^0$;

(b) or there is $k_0 \in J_{n-1}^0$ such that

$$\begin{cases} v_\vartheta^j(A) = v_\vartheta^j(B) & \text{for all } j \in J_{k_0-1}^0, \quad \text{and} \\ v_\vartheta^{k_0}(A) < v_\vartheta^{k_0}(B); \end{cases}$$

(c) or there is $k_0 \in J_{n-1}^0$ such that

$$\begin{cases} v_{\vartheta}^j(A) = v_{\vartheta}^j(B) & \text{for all } j \in J_{n-1}^0, \quad \text{and} \\ v_{\varphi_\alpha}^j(A) = v_{\varphi_\alpha}^j(B) & \text{for all } j \in J_{k_0-1}^0, \quad \text{and} \\ v_{\varphi_\alpha}^{k_0}(A) < v_{\varphi_\alpha}^{k_0}(B). \end{cases}$$

The previous conditions are incompatible, so it can only hold one of them. We point out that if $k_0 = 0$ in (b), then the condition "$v_{\vartheta}^j(A) = v_{\vartheta}^j(B)$ for all $j \in J_{k_0-1}^0$" is empty, and it only holds "$v_{\vartheta}^0(A) < v_{\vartheta}^0(B)$" (the same for case (c)). The binary relation $\overset{\alpha}{\leq}$ on $\mathcal{C}$ is called the *α-order on* $\mathcal{C}$. However, it is not an order, because it is not antisymmetric. In fact, it is a preference relation because it is transitive and total (so it is reflexive). When (a) holds, we write $A \overset{\alpha}{=} B$ and we say that A and B are *α-equal*. From $A \overset{\alpha}{=} B$ we cannot deduce $A = B$. When cases (b) or (c) hold, we write $A \overset{\alpha}{<} B$. Hence, $A \overset{\alpha}{\leq} B$ if and only if either $A \overset{\alpha}{=} B$ or $A \overset{\alpha}{<} B$.

Notice that the α-order $\overset{\alpha}{\leq}$ depends on a membership vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega_n$, a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in W_n$ and an aggregation function $\mathcal{A} : \bigcup_{m\in\mathbb{N}} \mathbb{R}^m \to \mathbb{R}$, but, for simplicity, we denote it simply by $\overset{\alpha}{\leq}$.

3. Some properties of the weight distribution functions

In this section, we explain how the first and the second weight distribution functions act on each weight vector. After that, we show some properties of both mappings. To start with, notice that ϑ does not depend on any membership vector α. It is only applied to a weight vector ω. By definition, $\vartheta^0(\omega) = \omega$. We now study ϑ.

Example 3.1. Let consider the vector

$$\omega = (\,0,\ 0.1,\ 0.15,\ 0.06,\ 0.2,\ 0.1,\ 0.23,\ 0.1,\ 0.06) \in W_9. \tag{2}$$

First of all, we observe that $0.1 + 0.15 + 0.06 + 0.2 + 0.1 + 0.23 + 0.1 + 0.06 = 1$, so ω is a weight vector. Some components of ω can be 0. In this case, such component will never change by applying ϑ. We determine the non-null minimum value on ω, which is:

$$\min\nolimits_+(\omega) = \min(\{\omega_1, \omega_2, ..., \omega_9\} \backslash \{0\}) = 0.06,$$

and we find the positions in ω whose value is 0.06, which are $\omega_4 = \omega_9 = 0.06$. The value 0 will be assigned by ϑ to such positions. The sum $\omega_4 + \omega_9 = 2 \cdot 0.06 = 0.12$ must be equally distributed on the rest of positions that are neither null nor 0.06. Therefore:

$$\begin{aligned} \vartheta(\omega) &= \left(0,\ 0.1 + \frac{0.12}{6},\ 0.15 + \frac{0.12}{6},\ 0,\ 0.2 + \frac{0.12}{6},\ 0.1 + \frac{0.12}{6},\ 0.23 + \frac{0.12}{6},\ 0.1 + \frac{0.12}{6},\ 0\right) \\ &= (0,\ 0.12,\ 0.17,\ 0,\ 0.22,\ 0.12,\ 0.25,\ 0.12,\ 0)\,. \end{aligned}$$

Notice that now $\vartheta(\omega)$ is another weight vector because $0.12 + 0.17 + 0.22 + 0.12 + 0.25 + 0.12 = 1$. The components of the vector $\vartheta(\omega)$ are denoted by

$$\vartheta(\omega) = (\,\vartheta(\omega)(1),\ \vartheta(\omega)(2), \ldots,\ \vartheta(\omega)(9)\,) = (0,\ 0.12,\ 0.17,\ 0,\ 0.22,\ 0.12,\ 0.25,\ 0.12,\ 0)\,.$$

Notice that as $\omega_1 = 0$, then $\vartheta(\omega)(1) = 0$. Hence $\vartheta(\omega)$ has strictly more zeros than ω (in fact, ω has one component zero and $\vartheta(\omega)$ has three of them). Next, we continue the process to determine $\vartheta^2(\omega)$. To do it, we determine:

$$\min\nolimits_+(\vartheta(\omega)) = \min(\{\vartheta(\omega)(1), \vartheta(\omega)(2), ..., \vartheta(\omega)(9)\} \backslash \{0\}) = 0.12,$$

and the components of $\vartheta(\omega)$ that take this value (which are $\vartheta(\omega)(2) = \vartheta(\omega)(6) = \vartheta(\omega)(8) = 0.12$). The sum $\vartheta(\omega)(2) + \vartheta(\omega)(6) + \vartheta(\omega)(8) = 3 \cdot 0.12 = 0.36$ must be equally distributed among the rest of non-null components, so:

$$\begin{aligned}\vartheta^2(\omega) &= \left(0,\, 0,\, 0.17 + \frac{0.36}{3},\, 0,\, 0.22 + \frac{0.36}{3},\, 0,\, 0.25 + \frac{0.36}{3},\, 0,\, 0\right)\\ &= (0,\, 0,\, 0.29,\, 0,\, 0.34,\, 0,\, 0.37,\, 0,\, 0)\,.\end{aligned}$$

The vector $\vartheta^2(\omega)$ is another weight vector because $0.29 + 0.34 + 0.37 = 1$. It has zeroes on six components. In fact, if $\vartheta(\omega)(j) = 0$ for some $j \in J_9$, then necessarily $\vartheta^2(\omega)(j) = 0$. In order to compute $\vartheta^3(\omega)$, we determine:

$$\min{}_+(\vartheta^2(\omega)) = \min(\{\vartheta^2(\omega)(1), \vartheta^2(\omega)(2), ..., \vartheta^2(\omega)(9)\}\backslash\{0\}) = 0.29,$$

which can only be found at position $j = 3$ on $\vartheta^2(\omega)$, so such quantity is equally distributed among the other non-null places. Hence:

$$\vartheta^3(\omega) = \left(0,\, 0,\, 0,\, 0,\, 0.34 + \frac{0.29}{2},\, 0,\, 0.37 + \frac{0.29}{2},\, 0,\, 0\right) = (0,\, 0,\, 0,\, 0,\, 0.485,\, 0,\, 0.515,\, 0,\, 0)\,.$$

Following the process, the value 0.485 is added to 0.515, so:

$$\vartheta^4(\omega) = \left(0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0.515 + \frac{0.485}{1},\, 0,\, 0\right) = (0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 1,\, 0,\, 0)\,.$$

Finally, as $\vartheta^4(\omega)(7) = 1$ is the unique non-null component, it is defined that

$$\vartheta^5(\omega) = (0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0) = \mathbf{0}_9,$$

so $\vartheta^k(\omega) = \mathbf{0}_9$ for all $k \geq 5$.

Taking into account the previous process, we can highlight the following facts.

1. The successive vectors

$$\begin{aligned}\omega &= (\,0,\, 0.1,\, 0.15,\, 0.06,\, 0.2,\, 0.1,\, 0.23,\, 0.1,\, 0.06),\\ \vartheta(\omega) &= (0,\, 0.12,\, 0.17,\, 0,\, 0.22,\, 0.12,\, 0.25,\, 0.12,\, 0)\,,\\ \vartheta^2(\omega) &= (0,\, 0,\, 0.29,\, 0,\, 0.34,\, 0,\, 0.37,\, 0,\, 0)\,,\\ \vartheta^3(\omega) &= (0,\, 0,\, 0,\, 0,\, 0.485,\, 0,\, 0.515,\, 0,\, 0)\,,\\ \vartheta^4(\omega) &= (0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 1,\, 0,\, 0)\end{aligned}$$

are linearly independent in $\mathbb{R}^9$ because the number of components with value zero is increasing, and when a component takes the value zero, the same component of the successive vectors are also zero. Hence, reordering the columns of the matrix of components, we can find a non-null minor of order five because it is associated to a triangular superior square submatrix, and the elements of its diagonal are non-null.

2. When we found the first moment k_0 in which $\vartheta^{k_0}(\omega) = \mathbf{0}_9$, then $\vartheta^k(\omega) = \mathbf{0}_9$ for all $k \geq k_0$. Then, the successive vectors $\{\vartheta^k(\omega) : k \geq k_0\}$ do not add dimension to the set $\{\omega, \vartheta(\omega), \vartheta^2(\omega), \ldots, \vartheta^{k_0-1}(\omega)\} \subset \mathbb{I}^9$.

In the next example we illustrate the action of the second distribution function φ_α.

Example 3.2. Let $\omega \in W_9$ be the weight vector given by (2), and let consider the associated membership vector:

$$\alpha = (\,0.2,\,0.75,\,1,\,0.2,\,0.5,\,0.9,\,0.23,\,0.75,\,0.2) \in \Omega_9.$$

All the membership degrees must be strictly positive. We now describe the action of φ_α. By definition, $\varphi_\alpha^0(\omega) = \omega$. Next we compute $\varphi_\alpha(\omega)$. As in the previous case, since $\omega_1 = 0$, then necessarily $\varphi_\alpha(\omega)(1) = 0$ (the zeroes are maintained). As in the computation of $\vartheta(\omega)$, we start determining:

$$\min_+(\omega) = \min(\{\omega_1, \omega_2, ..., \omega_9\} \backslash \{0\}) = 0.06,$$

and we observe the positions in ω whose value is 0.06, which are $\omega_4 = \omega_9 = 0.06$. However, in this case, before distributing $2 \cdot 0.06$ among the other non-null positions, it is necessary to observe the membership degrees associated to such positions, that is, $\{\alpha_4 = 0.2, \alpha_9 = 0.2\}$. Since these values are exactly the same, then $\varphi_\alpha(\omega)$ acts as $\vartheta(\omega)$, distributing $2 \cdot 0.06$ among the other non-null positions. Therefore,

$$\begin{aligned}\varphi_\alpha(\omega) &= \left(0,\,0.1 + \frac{0.12}{6},\,0.15 + \frac{0.12}{6},\,0,\,0.2 + \frac{0.12}{6},\,0.1 + \frac{0.12}{6},\,0.23 + \frac{0.12}{6},\,0.1 + \frac{0.12}{6},\,0\right)\\ &= (0,\,0.12,\,0.17,\,0,\,0.22,\,0.12,\,0.25,\,0.12,\,0) = \vartheta(\omega).\end{aligned}$$

Notice that $\varphi_\alpha(\omega)$ is another weight vector, and it respects the existing zeroes in ω. Next we repeat the process to compute $\varphi_\alpha^2(\omega)$. In this case,

$$\min_+(\varphi_\alpha(\omega)) = \min(\{\varphi_\alpha(\omega)(1), \varphi_\alpha(\omega)(2), ..., \varphi_\alpha(\omega)(9)\} \backslash \{0\}) = 0.12,$$

and the components of $\varphi_\alpha(\omega)$ that take this value are $\varphi_\alpha(\omega)(2) = \varphi_\alpha(\omega)(6) = \varphi_\alpha(\omega)(8) = 0.12$. Before distributing the sum $3 \cdot 0.12 = 0.36$ among the rest non-null components, we have to consider the associated membership degrees

$$\{\alpha_2 = 0.75,\,\alpha_6 = 0.9,\,\alpha_8 = 0.75\,\}\,.$$

In this case, such numbers are not the same, so we determine its minimum, which is 0.75, and the components in which we find such minimum (the second and the eighth). In this case, only the values $\varphi_\alpha(\omega)(2) = \varphi_\alpha(\omega)(8) = 0.12$ are distributed among the other components (but not $\varphi_\alpha(\omega)(6)$). Hence

$$\begin{aligned}\varphi_\alpha^2(\omega) &= \left(0,\,0,\,0.17 + \frac{0.24}{4},\,0,\,0.22 + \frac{0.24}{4},\,0.12 + \frac{0.24}{4},\,0.25 + \frac{0.24}{4},\,0,\,0\right)\\ &= (0,\,0,\,0.23,\,0,\,0.28,\,0.18,\,0.31,\,0,\,0)\,.\end{aligned}$$

Notice that $\varphi_\alpha^2(\omega) \neq \vartheta^2(\omega)$ because the vector α has been took into account. Now we repeat the process. The sixth component of $\varphi_\alpha(\omega)$ was 0.12, but this value were not distributed among the rest components because $\alpha_6 = 0.9 > 0.75 = \min\{0.75, 0.9\}$. Now the minimum is also the sixth component:

$$\min_+(\varphi_\alpha^2(\omega)) = \min(\{\varphi_\alpha^2(\omega)(1), \varphi_\alpha^2(\omega)(2), ..., \varphi_\alpha^2(\omega)(9)\} \backslash \{0\}) = 0.18,$$

which is reached at $\varphi_\alpha^2(\omega)(6) = 0.18$. As this is a unique value, we don't need to consider its associated membership degree $\alpha_6 = 0.9$ because we know that we are going to distribute the value $\varphi_\alpha^2(\omega)(6) = 0.18$ among the other non-null positions. Hence:

$$\begin{aligned}\varphi_\alpha^3(\omega) &= \left(0,\,0,\,0.23 + \frac{0.18}{3},\,0,\,0.28 + \frac{0.18}{3},\,0,\,0.31 + \frac{0.18}{3},\,0,\,0\right)\\ &= (0,\,0,\,0.29,\,0,\,0.34,\,0,\,0.37,\,0,\,0)\,.\end{aligned}$$

Notice that $\varphi_\alpha^3(\omega)=\vartheta^2(\omega)$. As the non-null values of $\varphi_\alpha^3(\omega)$ are distinct, on each step we don't need to look at their associated membership degrees, so the process continue as in the previous case:

$$\begin{aligned}
\varphi_\alpha^4(\omega) &= \left(0,\,0,\,0,\,0,\,0.34+\frac{0.29}{2},\,0,\,0.37+\frac{0.29}{2},\,0,\,0\right)=(0,\,0,\,0,\,0,\,0.485,\,0,\,0.515,\,0,\,0)\,;\\
\varphi_\alpha^5(\omega) &= \left(0,\,0,\,0,\,0,\,0,\,0,\,0.515+\frac{0.485}{1},\,0,\,0\right)=(0,\,0,\,0,\,0,\,0,\,0,\,1,\,0,\,0)\,;\\
\varphi_\alpha^k(\omega) &= \mathbf{0}_9 \quad \text{for all } k\geq 6.
\end{aligned}$$

Hence, to compute $\varphi_\alpha^k(\omega)$ it is necessary to know the membership vector α. Notice that, in this case, we have deduced that:

$$\begin{cases} \varphi_\alpha^j(\omega)=\vartheta^j(\omega) \quad \text{for all } j\in\{0,1\}, \qquad \text{and}\\ \varphi_\alpha^{j+1}(\omega)=\vartheta^j(\omega) \quad \text{for all } j\in\{2,3,\ldots\}.\end{cases}$$

Therefore, $\{\vartheta^k(\omega):k\in\mathbb{N}_0\}\subset\{\varphi_\alpha^j(\omega):k\in\mathbb{N}_0\}$, but $\varphi_\alpha^2(\omega)\notin\{\vartheta^k(\omega):k\in\mathbb{N}_0\}$.

After understanding the actions of ϑ and φ_α, the following properties (see [15]) are almost immediate for each $\omega\in W_n$ and all $\alpha\in\Omega_n$.

- Either $\vartheta(\omega)=\mathbf{0}_n$ or $\vartheta(\omega)$ is another weight vector.
- Either $\varphi_\alpha(\omega)=\mathbf{0}_n$ or $\varphi_\alpha(\omega)$ is another weight vector.
- If $\omega\neq\mathbf{0}_n$, then the vectors $\vartheta(\omega)$ and $\varphi_\alpha(\omega)$ have strictly more coordinates with value zero than the vector ω.
- If $k\in\{0,1,2,\ldots,n-1\}$, at least k components of $\vartheta^k(\omega)$ and $\varphi_\alpha^k(\omega)$ are zero.
- If $k\geq n$, then $\vartheta^k(\omega)=\varphi_\alpha^k(\omega)=\mathbf{0}_n$ for all $\omega\in W_n\cup\{\mathbf{0}_n\}$.

Reasoning as in the previous examples, we can state the following result.

Lemma 3.3. (cf. [19]) *The following properties hold.*

- *Given a weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)\in W_n$, there is a natural number $P_\omega\in\{1,2,\ldots,n\}$ such that the set $\{\,\vartheta^0(\omega),\vartheta^1(\omega),\ldots,\vartheta^{P_\omega-1}(\omega)\,\}$ is linearly independent in $\mathbb{R}^n$ and $\vartheta^k(\omega)=\mathbf{0}_n\in\mathbb{R}^n$ for all $k\geq P_\omega$.*
- *Given a membership vector $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and a weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)\in W_n$, there is a natural number $Q_{\alpha,\omega}\in\{1,2,\ldots,n\}$ such that the set $\{\,\varphi_\alpha^0(\omega),\varphi_\alpha^1(\omega),\ldots,\varphi_\alpha^{Q_{\alpha,\omega}-1}(\omega)\,\}$ is linearly independent in $\mathbb{R}^n$ and $\varphi_\alpha^k(\omega)=\mathbf{0}_n\in\mathbb{R}^n$ for all $k\geq Q_{\alpha,\omega}$.*

Next we introduce a case in which the whole set $\{\,\vartheta^0(\omega),\vartheta^1(\omega),\ldots,\vartheta^{n-1}(\omega)\,\}$ is linearly independent and a concrete choice of α (concretely, $\alpha=\mathbf{1}_n$) in which all the scalar products of the previous vectors with α produce the same non zero real number.

Proposition 3.4. *Given a weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)\in W_n$ such that $0<\omega_i\neq\omega_j$ for all $i,j\in J_n$ with $i\neq j$, the set*

$$\{\,\vartheta^0(\omega),\vartheta^1(\omega),\ldots,\vartheta^{n-1}(\omega)\,\}$$

is a basis of $\mathbb{R}^n$. Furthermore, if $\alpha=\mathbf{1}_n$, then

$$\langle\vartheta^k(\omega),\alpha\rangle=1 \quad \textit{for all } k\in J_{n-1}^0. \tag{3}$$

Proof. Clearly $\vartheta^0(\omega) = \omega$. By hypothesis, all its components are distinct to zero. Let $i_0 \in J_n$ be the unique index such that $\omega_{i_0} = \min_+(\omega)$ (it is unique because each two distinct components of ω are distinct). Then the components of $\vartheta^1(\omega)$ are:

$$\vartheta^1(\omega)(j) = \begin{cases} 0, & \text{if } j = i_0, \\ \omega_j + \dfrac{\omega_{i_0}}{n-1}, & \text{if } j \neq i_0. \end{cases}$$

Notice that the vector $\vartheta^1(\omega)$ has a unique component which is zero (precisely, the i_0-component), and the others have been increased by the same constant (which is $\omega_{i_0}/(n-1)$), so each two distinct non-null components of $\vartheta^1(\omega)$ take distinct values. And this process continues: $\vartheta^2(\omega)$ has exactly two components taking the value zero (i_0 and i_1), and the others components are distinct between them. Repeating this procedure, as the number of null components increases one by one, then we get a set of vectors $\{\, \vartheta^0(\omega), \vartheta^1(\omega), \ldots, \vartheta^{n-1}(\omega)\,\}$ whose matrix of components is equivalent (unless reordering the columns) to a triangular and superior matrix with non-null principal diagonal, so its vectors are linearly independent. Hence, it is a basis of $\mathbb{R}^n$.

Furthermore, if $\alpha = \mathbf{1}_n$, since

$$\langle \vartheta^k(\omega), \mathbf{1} \rangle = \sum_{j=0}^{n-1} \left[\vartheta^k(\omega)(j) \cdot 1 \right] = \sum_{j=0}^{n-1} \vartheta^k(\omega)(j)$$

is the sum of components of the vector $\vartheta^k(\omega)$, and it is a weight vector, then $\langle \vartheta^k(\omega), \mathbf{1} \rangle = 1$ for all $k \in J^0_{n-1}$. □

Notice that equations (3) are not incompatible even if $\{\, \vartheta^0(\omega), \vartheta^1(\omega), \ldots, \vartheta^{n-1}(\omega)\,\}$ is a basis of $\mathbb{R}^n$. For instance, if $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{R}^n$, then $\langle \mathbf{e}_j, \mathbf{1}_n \rangle = 1$ for all $j \in J_n$.

In the next result, we compare the number of zeros of ω, $\vartheta(\omega)$ and $\varphi_\alpha(\omega)$. To do it, we use the following notation. Given a finite and non-empty set X, let denote by $\operatorname{card}(X)$ to the *cardinal* of X (that is, its number of elements). We will distinguish between the cardinal of X and the *cardinal of X avoiding repetitions*, which is the number of distinct elements in X. For instance, the cardinal of $X = \{\, \omega_1 = 0.1, \omega_2 = 0.1, \omega_3 = 0.1, \omega_4 = 0.7\,\}$ is $\operatorname{card}(X) = 4$, but the cardinal of X avoiding repetitions is 2 because the only distinct points of X are 0.1 and 0.7. Accordingly, given a vector $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$, let denote by $\operatorname{card}_0(\mathbf{u})$ to the number of coordinates of $\mathbf{u}$ that takes the value zero, that is, $\operatorname{card}_0(\mathbf{u}) = \operatorname{card}(\{\, j \in J_m : u_j = 0\,\})$.

Proposition 3.5. *Given a membership vector α and a weight vector $\omega \in W_n$,*

$$\operatorname{card}_0(\omega) < \operatorname{card}_0(\varphi_\alpha(\omega)) \leq \operatorname{card}_0(\vartheta(\omega)). \tag{4}$$

Proof. Let use the notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, $\vartheta(\omega) = (\omega^1_1, \omega^1_2, \ldots, \omega^1_n)$ and $\varphi_\alpha(\omega) = (\gamma^1_1, \gamma^1_2, \ldots, \gamma^1_n)$. By definition, the mapping ϑ considers the positions $\{j_1, j_2, \ldots, j_p\} \subseteq J_n$ (with $p \geq 1$) for which $\omega_j = \min_+(\omega) > 0$, and directly defines $\omega^1_j = \vartheta(\omega)(j) = 0$ for all $j \in \{j_1, j_2, \ldots, j_p\}$. Then, the sum $\omega_{j_1} + \omega_{j_2} + \ldots + \omega_{j_p}$ is equally distributed through the rest of strictly positive coordinates of ω. Therefore, using this notation, the vector $\vartheta(\omega)$ has p zeros more than ω, that is, $\operatorname{card}_0(\vartheta(\omega)) = \operatorname{card}_0(\omega) + p$. Then $\operatorname{card}_0(\vartheta(\omega)) > \operatorname{card}_0(\omega)$. However, φ_α acts in a slightly different way: after computing the positions $\{j_1, j_2, \ldots, j_p\} \subseteq \{1, 2, \ldots, n\}$ for which $\omega_j = \min_+(\omega) > 0$, it must consider the indices $j \in \{j_1, j_2, \ldots, j_p\}$ for which $\alpha_j = \min(\{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_p}\})$. Then, there are $\{j'_1, j'_2, \ldots, j'_{p'}\} \subseteq \{j_1, j_2, \ldots, j_p\}$, with $1 \leq p' \leq p$, for which $\alpha_{j'} = \min(\{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_p}\})$. The mapping φ_α defines $\gamma^1_{j'} = \varphi_\alpha(\omega)(j') = 0$ for all $j' \in \{j'_1, j'_2, \ldots, j'_{p'}\}$, so the vector $\varphi_\alpha(\omega)$ has p' zeros more than ω (that is, $\operatorname{card}_0(\varphi_\alpha(\omega)) = \operatorname{card}_0(\omega) + p'$). Since $1 \leq p' \leq p$, then

$$\operatorname{card}_0(\omega) < \operatorname{card}_0(\omega) + p' = \operatorname{card}_0(\varphi_\alpha(\omega)) \leq \operatorname{card}_0(\omega) + p = \operatorname{card}_0(\vartheta(\omega)),$$

so (4) holds. □

4. Some properties of the α-order

In this section, we show a simple way to apply the α-order by building suitable tables, even in the case of considering fuzzy sets that are not fuzzy numbers. We take advantage of the description we have done about the first and second weight distribution functions. Finally, we present some new proprties of the α-order.

Example 4.1. Suppose that A_1, A_2 and A_3 are the fuzzy sets of $\mathbb{R}$ whose membership functions are given, for each $t \in \mathbb{R}$, by:

$$A_1(t) = \begin{cases} \dfrac{-1}{360} \begin{pmatrix} 5t^8 - 127t^7 + 1355t^6 - 7906t^5 + 27545t^4 \\ -58747t^3 + 75315t^2 - 53640t + 16200 \end{pmatrix}, & \text{if } 1 \leq t \leq 5, \\ 0, & \text{otherwise;} \end{cases}$$

$$A_2(t) = (\,1\,/\,2.5\,/\,3.5\,/\,5.5\,)(t) = \begin{cases} \dfrac{t-1}{1.5}, & \text{if } 1 < t < 2.5, \\ 1, & \text{if } 2.5 \leq t \leq 3.5, \\ \dfrac{5.5-t}{2}, & \text{if } 3.5 < t < 5.5, \\ 0, & \text{otherwise;} \end{cases}$$

$$A_3(t) = \begin{cases} 1, & \text{if } t \in \{3, 3.5\}, \\ 0.7, & \text{if } t \in \{2, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

The membership functions of A_1, A_2 and A_3 are plotted in Figure 2 (A_1 in blue, A_2 in green and A_3 in red; for A_3 they are only depicted the non-null values).

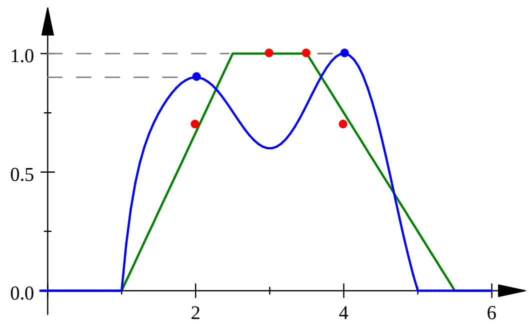


Figure 2: Graphic representation of the fuzzy sets of Example 4.1.

The fuzzy set A_2 is a trapezoidal fuzzy number, but A_1 and A_3 are fuzzy sets of $\mathcal{C}$ that are not fuzzy numbers. Lets apply to them the α-order by employing the vectors:

$$\alpha = (1, 0.9, 0.9, 0.75) \in \Omega_4 \quad \text{and} \quad \omega = (0.5, 0.2, 0.15, 0.15) \in W_4.$$

This choice means that we are mainly interested on 1-cuts, 0.9-cuts and 0.75-cuts to carry out the α-ordering. Furthermore, we associated a weight of 0.5 to 1-cuts, 0.2 to 0.9-cuts at the first time, 0.15 to 0.9-cuts at the second time, and 0.1 to 0.75-cuts. Obviously ω is a weight vector because $0.5 + 0.2 + 0.15 + 0.15 = 1$. We approximately compute the corresponding α-cuts of A_1, A_2 and A_3 (see Table 1).

	A_1	A_2	A_3
1-cut	$\{4\}$	$[2.5, 3.5]$	$\{3\} \cup \{3.5\}$
0.9-cut	$\{2\} \cup [3.691, 4.268]$	$[2.35, 3.7]$	$\{3\} \cup \{3.5\}$
0.75-cut	$[1.503, 2.509] \cup [3.445, 4.425]$	$[2.125, 4]$	$\{2\} \cup \{3\} \cup \{3.5\} \cup \{4\}$

Table 1: Distinct α-cuts of the fuzzy sets of Example 4.1.

Successively applying the weight distribution functions to ω as we have explained on Section 3, we obtain the following vectors:

$$\begin{aligned}
\vartheta^0(\omega) &= \varphi_\alpha^0(\omega) = \omega = (0.5, 0.2, 0.15, 0.15),\\
\varphi_\alpha^1(\omega) &= (0.55, 0.25, 0.2, 0),\\
\vartheta^1(\omega) &= \varphi_\alpha^2(\omega) = (0.65, 0.35, 0, 0),\\
\vartheta^2(\omega) &= \varphi_\alpha^3(\omega) = (1, 0, 0, 0),\\
\vartheta^k(\omega) &= \varphi_\alpha^{k+1}(\omega) = \mathbf{0}_4 \qquad \text{for all } k \geq 3.
\end{aligned} \tag{5}$$

Let consider the *mean aggregation function*, that is,

$$\overline{M}(t_1, t_2, \ldots, t_m) = \frac{t_1 + t_2 + \ldots + t_m}{m} \quad \text{for each } m \in \mathbb{N} \text{ and each } t_1, t_2, \ldots, t_m \in \mathbb{R}.$$

To apply the α-order, we use the following formulae:

$$v_\vartheta^0(A_i) = \langle\, \vartheta^0(\omega), \mathcal{A}_\alpha^{A_i} \,\rangle = \langle\, \omega, \mathcal{A}_\alpha^{A_i} \,\rangle = \sum_{j=1}^{n} \left[\omega_j \cdot \overline{M}_{\alpha_j}^{A_i} \right]$$

to compute Table 2.

A_i	j	α_j	$\mathbb{I}_{\alpha_j}^{A_i}$	$\overline{M}_{\alpha_j}^{A_i}$	ω_j	$v_\vartheta^0(A_i)$
A_1	1	1	$\{4\}$	4	0.5	3.608
	2	0.9	$\{2\} \cup [3.691, 4.268]$	3.320	0.2	
	3	0.9	$\{2\} \cup [3.691, 4.268]$	3.320	0.15	
	4	0.75	$[1.503, 2.509] \cup [3.445, 4.425]$	2.971	0.15	
A_2	1	1	$[2.5, 3.5]$	3	0.5	3.018
	2	0.9	$[2.35, 3.7]$	3.025	0.2	
	3	0.9	$[2.35, 3.7]$	3.025	0.15	
	4	0.75	$[2.125, 4]$	3.063	0.15	
A_3	1	1	$\{3\} \cup \{3.5\}$	3.25	0.5	3.231
	2	0.9	$\{3\} \cup \{3.5\}$	3.25	0.2	
	3	0.9	$\{3\} \cup \{3.5\}$	3.25	0.15	
	4	0.75	$\{2\} \cup \{3\} \cup \{3.5\} \cup \{4\}$	3.125	0.15	

Table 2: Computation of 0-valuations of the fuzzy sets of Example 4.1.

The values $v_\vartheta^0(A_1) = 3.608$, $v_\vartheta^0(A_2) = 3.018$ and $v_\vartheta^0(A_3) = 3.231$ are enough to α-order the fuzzy sets A_1, A_2 and A_3. In this case, as $v_\vartheta^0(A_2) < v_\vartheta^0(A_3) < v_\vartheta^0(A_1)$, we lead to the α-ordering:

$$A_2 \overset{\alpha}{<} A_3 \overset{\alpha}{<} A_1.$$

If we have obtained that $v_\vartheta^0(A_{j_1}) = v_\vartheta^0(A_{j_2})$ for some $j_1, j_2 \in \{1,2,3\}$, then this criteria would have not been sufficient to decide whether $A_{j_1} \overset{\alpha}{<} A_{j_2}$, $A_{j_1} \overset{\alpha}{>} A_{j_2}$ or $A_{j_1} \overset{\alpha}{=} A_{j_2}$. In such a case, we would have resorted to the second criteria, by using Table 3, in which we write the components of the vector $\vartheta(\omega)$ rather than of the vector ω.

A_i	$\overline{M}_{\alpha_j}^{A_i}$	$\vartheta(\omega)(j)$	$v_\vartheta^1(A_i)$
A_1	4	0.65	3.762
	3.320	0.35	
	3.320	0	
	2.971	0	
A_2	3	0.65	3.009
	3.025	0.35	
	3.025	0	
	3.063	0	
A_3	3.25	0.65	3.25
	3.25	0.35	
	3.25	0	
	3.125	0	

Table 3: Computation of 1-valuations ($v_\vartheta^1(A_i) = \sum_{j=1}^n [\vartheta(\omega)(j) \cdot \overline{M}_{\alpha_j}^{A_i}]$) of the fuzzy sets of Example 4.1.

If it would have been necessary, we have compared the values $\{v_\vartheta^1(A_i)\}_{i=1}^3$ to get the α-ordering for the fuzzy sets for which, in the first step, we have obtained $v_\vartheta^0(A_{j_1}) = v_\vartheta^0(A_{j_2})$. If there were more ties, the process would have continued with the vector $\vartheta^2(\omega)$, and, after that, if necessary, with the vectors $\{\varphi_\alpha^1(\omega), \varphi_\alpha^2(\omega), \varphi_\alpha^3(\omega)\}$. Notice that it is not necessary to come back to $\varphi_\alpha^0(\omega)$ because $\varphi_\alpha^0(\omega) = \vartheta^0(\omega) = \omega$, so such step can be omitted.

The α-order produces reasonable orderings among fuzzy numbers because it is an admissible preorder (see [16, Theorem 4.1]). Therefore, it refines the Klir and Yuan order $\preceq_{KY}$ on the set of all fuzzy numbers of the real line, which is a very intuitive order from the human point of view (the main problem of the Klir and Yuan order $\preceq_{KY}$ is that it is not total).

In the following result, we describe how obtain the successive real values when applying the α-order in the following case: the aggregation function is the mean and the fuzzy sets are triangular fuzzy numbers. We come back to the notation $v_{\alpha,\omega,\vartheta,\overline{M}}^k$ because it is important to especify that the aggregation function is $\overline{M}$.

Theorem 4.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega_n$ be a membership vector, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in W_n$ be a weight vector and let $\mathcal{A} = \overline{M}$ be the mean aggregation function. Then, for each $k \in J_{n-1}^0$, the k-valuations* *of any triangular fuzzy number* $A = (a_1/a_2/a_3) \in \mathrm{TFN}(\mathbb{R})$ *are:*

$$\begin{aligned} v^k_{\alpha,\omega,\vartheta,\overline{M}}(A) &= \left\langle \vartheta^k(\omega), \frac{a_1+a_3}{2}(\mathbf{1}_n - \alpha) + a_2\alpha \right\rangle \\ &= \begin{cases} \dfrac{a_1+a_3}{2} + \left(a_2 - \dfrac{a_1+a_3}{2}\right) \langle \vartheta^k(\omega), \alpha\rangle, & \text{if } \vartheta^k(\omega) \neq \mathbf{0}_n, \\ 0, & \text{if } \vartheta^k(\omega) = \mathbf{0}_n, \end{cases} \end{aligned} \tag{6}$$

and

$$\begin{aligned} v^k_{\alpha,\omega,\varphi_\alpha,\overline{M}}(A) &= \left\langle \varphi^k_\alpha(\omega), \frac{a_1+a_3}{2}(\mathbf{1}_n - \alpha) + a_2\alpha \right\rangle \\ &= \begin{cases} \dfrac{a_1+a_3}{2} + \left(a_2 - \dfrac{a_1+a_3}{2}\right) \langle \varphi^k_\alpha(\omega), \alpha\rangle, & \text{if } \varphi^k_\alpha(\omega) \neq \mathbf{0}_n, \\ 0, & \text{if } \varphi^k_\alpha(\omega) = \mathbf{0}_n. \end{cases} \end{aligned}$$

Proof. If $\vartheta^k(\omega) = \mathbf{0}_n$, then clearly $v^k_{\alpha,\omega,\vartheta,\overline{M}}(A) = 0$, and if $\varphi^k_\alpha(\omega) = \mathbf{0}_n$, then $v^k_{\alpha,\omega,\varphi_\alpha,\overline{M}}(A) = 0$. We study each case separately by considering that $\vartheta^k(\omega) \neq \mathbf{0}_n$ and/or $\varphi^k_\alpha(\omega) \neq \mathbf{0}_n$. In such a case, given $\beta \in (0,1]$, since $A = (a_1/a_2/a_3) \in \mathrm{TFN}(\mathbb{R})$, its β-cuts is:

$$A_\beta = [\, a_1 + (a_2 - a_1)\beta,\, a_3 - (a_3 - a_2)\,\beta\,].$$

Therefore,

$$\begin{aligned} \overline{M}^A_\beta &= \overline{M}\,(a_1 + (a_2 - a_1)\beta,\, a_3 - (a_3 - a_2)\,\beta) = \frac{a_1 + (a_2 - a_1)\beta + a_3 - (a_3 - a_2)\,\beta}{2} \\ &= \frac{a_1+a_3}{2} + \beta\left(a_2 - \frac{a_1+a_3}{2}\right). \end{aligned}$$

In particular, taking into account that the sum of components of the vector

$$\vartheta^k(\omega) = \Big(\, \vartheta^k(\omega)(1),\, \vartheta^k(\omega)(2), \ldots,\, \vartheta^k(\omega)(n) \Big) \in W_n$$

is 1 (because it is a non-null weight vector), we deduce that, for all $k \in J^0_{n-1}$:

$$\begin{aligned} v^k_{\alpha,\omega,\vartheta,\overline{M}}(A) &= \sum_{j=1}^n \left[\vartheta^k(\omega)(j) \cdot \overline{M}^A_{\alpha_j} \right] = \sum_{j=1}^n \left[\vartheta^k(\omega)(j) \cdot \left(\frac{a_1+a_3}{2} + \alpha_j \left(a_2 - \frac{a_1+a_3}{2} \right) \right) \right] \\ &= \sum_{j=1}^n \left[\vartheta^k(\omega)(j) \cdot \frac{a_1+a_3}{2} \right] + \sum_{j=1}^n \left[\vartheta^k(\omega)(j) \cdot \alpha_j \left(a_2 - \frac{a_1+a_3}{2} \right) \right] \\ &= \frac{a_1+a_3}{2} \sum_{j=1}^n \left[\vartheta^k(\omega)(j) \right] + \left(a_2 - \frac{a_1+a_3}{2} \right) \sum_{j=1}^n \left[\vartheta^k(\omega)(j) \cdot \alpha_j \right] \\ &= \frac{a_1+a_3}{2} + \left(a_2 - \frac{a_1+a_3}{2} \right) \langle \vartheta^k(\omega), \alpha \rangle. \end{aligned}$$

Notice that:

$$\begin{aligned}
&\left\langle \vartheta^k(\omega), \frac{a_1+a_3}{2}(\mathbf{1}_n-\alpha)+a_2\alpha \right\rangle \\
&\quad= \left\langle \vartheta^k(\omega), \frac{a_1+a_3}{2}(1-\alpha_1,1-\alpha_2,\dots,1-\alpha_n)+a_2(\alpha_1,\alpha_2,\dots,\alpha_n) \right\rangle \\
&\quad= \sum_{j=1}^n \left[\vartheta^k(\omega)(j)\cdot\left(\frac{a_1+a_3}{2}\,(1-\alpha_j)+a_2\,\alpha_j\right)\right] \\
&\quad= \sum_{j=1}^n \left[\vartheta^k(\omega)(j)\cdot\left(\frac{a_1+a_3}{2}+\left(a_2-\frac{a_1+a_3}{2}\right)\alpha_j\right)\right] \\
&\quad= \frac{a_1+a_3}{2}\sum_{j=1}^n \left[\vartheta^k(\omega)(j)\right]+\left(a_2-\frac{a_1+a_3}{2}\right)\sum_{j=1}^n\left[\vartheta^k(\omega)(j)\,\cdot\alpha_j\right] \\
&\quad= v^k_{\alpha,\omega,\vartheta,\overline{M}}(A).
\end{aligned}$$

These equalities prove that (6) holds. Replacing $\vartheta^k(\omega)$ by $\varphi^k_\alpha(\omega)$, we deduce the same for $v^k_{\alpha,\omega,\varphi_\alpha,\overline{M}}(A)$. □

The previous result shows an easy way to compute the k-valuations of any triangular fuzzy number. Next, we illustrate such result.

Example 4.3. Let α and ω the vectors given in Example 4.1. In (5), there were computed the vectors $\vartheta^k(\omega)$ and $\varphi^k_\alpha(\omega)$ for any $k\in\mathbb{N}_0$. Let consider the triangular fuzzy number $A=(1/5/7)$. Since $a_2=5$ and $(a_1+a_3)/2=4$, then, applying (6), for each $k\in J^0_2$:

$$\begin{aligned}
v^k_{\alpha,\omega,\vartheta,\overline{M}}(A) &= \frac{a_1+a_3}{2}+\left(a_2-\frac{a_1+a_3}{2}\right)\langle\vartheta^k(\omega),\alpha\rangle = 4+(5-4)\langle\,\vartheta^k(\omega),(1,0.9,0.9,0.75)\,\rangle \\
&= \left\{\begin{array}{ll} 4+\langle\,(0.5,0.2,0.15,0.15),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=0,\\ 4+\langle\,(0.65,0.35,0,0),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=1,\\ 4+\langle\,(1,0,0,0),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=2 \end{array}\right\} \\
&= \left\{\begin{array}{ll} 4.9275, & \text{if } k=0,\\ 4.965, & \text{if } k=1,\\ 5, & \text{if } k=2. \end{array}\right.
\end{aligned}$$

If $k\geq 3$, since $\vartheta^k(\omega)=\mathbf{0}_4$, then $v^k_{\alpha,\omega,\vartheta,\overline{M}}(A)=0$. Regarding the second group of valuations, for each $k\in J^0_3$:

$$\begin{aligned}
v^k_{\alpha,\omega,\varphi_\alpha,\overline{M}}(A) &= \frac{a_1+a_3}{2}+\left(a_2-\frac{a_1+a_3}{2}\right)\langle\varphi^k_\alpha(\omega),\alpha\rangle = 4+(5-4)\langle\,\varphi^k_\alpha(\omega),(1,0.9,0.9,0.75)\,\rangle \\
&= \left\{\begin{array}{ll} 4+\langle\,(0.5,0.2,0.15,0.15),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=0,\\ 4+\langle\,(0.55,0.25,0.2,0),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=1,\\ 4+\langle\,(0.65,0.35,0,0),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=2,\\ 4+\langle\,(1,0,0,0),\ (1,0.9,0.9,0.75)\,\rangle, & \text{if } k=3 \end{array}\right\} \\
&= \left\{\begin{array}{ll} 4.9275, & \text{if } k=0,\\ 4.955 & \text{if } k=1,\\ 4.965, & \text{if } k=2,\\ 5, & \text{if } k=3, \end{array}\right.
\end{aligned}$$

and similarly $v^k_{\alpha,\omega,\varphi_\alpha,\overline{M}}(A)=0$ for all $k\geq 4$.

5. Conclusions and prospect work

In this paper, we have described how it can be applied the α-order to any fuzzy sets in the class $\mathcal{C}$. Before that, we have introduced the algebraic tools that are underlying to this construction, and we have detailed the class of fuzzy sets of the real line that can be ranked by using the mentioned binary relation. Furthermore, we have introduced some new properties both of the weight distribution functions and of the α-order. To complete the paper, we have illustrated all processes by showing examples enough.

The α-order verifies good properties, especially according to human intuition. Hence, its study is an open field that deserves to be explored. There are necessary to deep in the algebraic characteristics of such binary relation, a to develop an easy computational tool to carry out the ordering from a automatic point of view.

Acknowledgments. The authors acknowledge with thanks to their universities. A.F. Roldán López de Hierro and H. Bustince are grateful to Ministerio de Ciencia e Innovación by Projects PID2020-119478GB-I00 and PID2022-136627NB-I00, supported by MCIN/AEI/10.13039/501100011033/FEDER, UE. The first author is also grateful to FEDER 2021-2027/Junta de Andalucía-Consejería de Universidad, Investigación e Innovación, project C-EXP-153-UGR23.

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