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# Fixed points of order preserving contractions

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**Abstract**

In this paper, we analyze fixed points of order preserving contractions on ordered metric spaces. Fixed point property of such maps is characterized for bounded convex subsets of the Euclidean plane.

*Keywords:* Contraction, order preserving map, fixed point property
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## 1. Introduction

Let $X = [0, 1) \cup (2, 3]$. There exist fixed-point-free contractions of $X$ into $X$, and there exist fixed-point-free order preserving mappings of $X$ into $X$; but every mapping on $X$ that is both a contraction and order preserving has a fixed point. This paper identifies subsets of a partially ordered metric space on which every order preserving contraction has a fixed point.

Fixed point theorems were first introduced in topological spaces for continuous functions and it is now developed as a separate branch of mathematics. Existence of fixed points is of significant importance and over the past 50 years, it has been revealed as a very powerful and important tool in the study of nonlinear mathematical structures. Fixed point property for continuous functions on topological spaces, contractions on metric spaces and order preserving functions on ordered sets have been separately investigated by many researchers ([1, 3, 9, 10]). Many applications exist for theorems on sets $X$ and functions $f : X \to X$ if $f$ is either a continuous function on the topological space $X$ or if $f$ is a contraction on the metric space $X$ or if

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$f$ is order preserving on the partially ordered set $X$. Fixed points of order preserving continuous functions on ordered metric spaces have been investigated by [2, 5, 7, 8].

In order to distinguish between various fixed point properties on different mathematical structures, we use the following terminologies. If every order preserving function on a partially ordered set $X$ fixes a point, then we say that $X$ has the *order fixed point property* or in short *order-FPP*. If every contraction on a metric space $X$ fixes a point, then we say that $X$ has the *contraction fixed point property* or in short *contraction-FPP*.

Two elements $x, y$ in a poset $(P, \leq)$ are said to be *comparable* if either $x \leq y$ or $y \leq x$. If $C$ is a subset of $P$ and any two elements of $C$ are comparable, then $C$ is called a *chain*. If every non-empty chain of $P$ has both supremum and infimum in $P$, then we say that $P$ is *doubly chain-complete*. $X = (X, d, \leq)$ is called an *ordered metric space* if $(X, d)$ is a metric space and $(X, \leq)$ is a partially ordered set.

## 2. FPP for order preserving contractions on ordered metric spaces

We now introduce a new type of fixed point property for ordered metric spaces as follows.

**Definition 2.1.** *Let $X$ be an ordered metric space. If every order preserving contraction $f : X \to X$ has a fixed point, then $X$ is said to have the* crop fixed point property*; in short* crop-FPP. *An order preserving contraction $f : X \to X$ that does not fix any point is called a* crop fixed-point-free map.

**Example 2.2.** *Let $X = [0,\ 1) \cup (2,\ 3]$.*
*Then $X$ does not have the order-FPP as $g : X \to X$ defined by*

$$g(x) = \begin{cases} \frac{x+1}{2} & \text{if } x \in [0,\ 1) \\ \frac{x+2}{2} & \text{if } x \in (2,\ 3]. \end{cases}$$

*is an order preserving map that does not fix any element of $X$.*

*Further, the map $h : X \to X$ defined by*

$$h(x) = \begin{cases} \frac{x+1}{2} & \text{if } x \in [0,\ 1) \\ \frac{4-x}{2} & \text{if } x \in (2,\ 3]. \end{cases}$$

*is a contraction that does not fix any element of $X$. Thus $X$ does not have the contraction-FPP.*

*Let $f : X \to X$ be any order preserving contraction, with contraction constant $\alpha$.*
*For $\varepsilon > 0$, sufficiently small $1 - \varepsilon$ and $2 + \varepsilon$ are in $X$.*

$$\begin{aligned} |f(2+\varepsilon) - f(1-\varepsilon)| &\leq \alpha|(2+\varepsilon) - (1-\varepsilon)| \\ &\leq \alpha(1+2\varepsilon) \\ &< 1 \text{ for } \varepsilon \text{ sufficiently small} \end{aligned}$$

*Thus either both $f(1-\varepsilon)$ and $f(2+\varepsilon)$ are in $[0,\ 1)$ or in $(2,\ 3]$, say the first. Since both $[0,\ 1)$ and $(2,\ 3]$ are connected and $f : X \to X$ is continuous, we have $f(X) \subseteq [0,\ 1)$.*

*Thus the order preserving contraction $f$ takes $[0,\ f(3)]$ to itself and hence by Tarski's theorem [10] (or by Banach's contraction principle [1]), $f$ fixes an element of $X$. Thus $X$ has the crop-FPP.*

*Hence $X$ has neither the order-FPP nor the contraction-FPP, but has the crop-FPP.*

**Definition 2.3** ([4])**.** *Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces. A function $f : X \to Y$ is called a* similitude *if there exists $r > 0$ such that $d_2(f(x), f(y)) = r.d_1(x, y)\ \ \forall x, y \in X$. In this case, we say that $X$ and $Y$ are similar.*

**Definition 2.4.** *Two ordered metric spaces $X$ and $Y$ are* isotonically similar *if there exists an order preserving (isotone) similitude map $f$ from $X$ onto $Y$.*

**Theorem 2.5.** *Let $X$ and $Y$ be isotonically similar ordered metric spaces. Then $X$ has the crop-FPP if and only if $Y$ has the crop-FPP.*

*Proof.* Let $f : X \to Y$ be an onto map that is both order preserving and similitude. Suppose $X$ has the crop-FPP. Let $g : Y \to Y$ be any order preserving contraction, with contraction constant $\alpha$. Since $f$ is a similitude it is one-one and hence $f$ is bijective. Thus $f^{-1}$ is also an order preserving similitude. Hence $f^{-1} \circ g \circ f$ is an order preserving contraction on $X$ with contraction constant $\alpha$. Since $X$ has the crop-FPP, there exists $x \in X$ such that $f^{-1} \circ g \circ f(x) = x$. But then $g(f(x)) = f(x)$ so that $g$ fixes the element $f(x) \in Y$. This proves that $Y$ has the crop-FPP. On the other hand, crop-FPP of $X$ follows from the crop-FPP of $Y$ as isotonically similar is an equivalence relation. □

**Definition 2.6.** *Let $(X, d, \leq)$ be an ordered metric space. An order preserving map $r : X \to X$ is called a* crop-retraction *if $d(r(x), r(y)) \leq d(x, y)$ for all $x, y \in X$ and $r \circ r = r$. In this case $r(X)$ is called a* crop-retract *of $X$.*

**Theorem 2.7.** *If $X$ has the crop-FPP, then every crop-retract of $X$ has the crop-FPP.*

*Proof.* Let $S$ be a crop-retract of $X$. Then there exists a crop-retraction $r : X \to X$ such that $r(X) = S$. Let $g : S \to S$ be any order preserving contraction. Then $g \circ r : X \to X$ is an order preserving contraction. By the assumption, there exists an element $x_0 \in X$ such that $g \circ r(x_0) = x_0$. As $x_0 = g(r(x_0)) \in S$, $r(x_0) = x_0$ so that $g(x_0) = x_0$. This proves that $S$ has the crop-FPP.

□

## 3. crop-FPP for Subsets of Euclidean space

The Euclidean space $\mathbb{R}^n$ is the most simple ordered metric space, which is induced with the usual metric and point-wise ordering. We shall now analyze crop-FPP for convex subsets of $\mathbb{R}$ and $\mathbb{R}^2$; a subset $X$ of $\mathbb{R}^n$ is convex in the sense that for every pair $x, y \in X$, the line segment joining $x$ and $y$ lies in $X$.

**Lemma 3.1.** *Let $C$ be any convex subset of $\mathbb{R}^n$ and $c \in C$. Then $C - \{c\}$ does not have the crop-FPP*

*Proof.* The map $f : C - \{c\} \to C - \{c\}$ defined by $f(x) = \frac{1}{2}(x + c)$ is an order preserving contraction, that does not fix any element of $C - \{c\}$. This proves that $C - \{c\}$ does not have the crop-FPP. □

### *3.1. crop-FPP for subsets of the real line $\mathbb{R}$*

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The following two theorems characterize crop-FPP for convex subsets of the real line.

**Theorem 3.2.** *A convex set $P$ in $\mathbb{R}$ has the crop-FPP if and only if $P$ is a closed set.*

*Proof.* If $P$ is closed, then it is complete and hence by contraction principle, $P$ has the contraction-FPP. Thus $P$ has the crop-FPP. Conversely, if $P$ is not closed, then there is a limit point $a$ of $P$ that does not lie in $P$. In this case $P \cup \{a\}$ is convex and hence by Lemma 3.1, $P$ does not have the crop-FPP. □

For a bounded convex set $P$ in $\mathbb{R}$, crop-FPP and order-FPP coincide.

**Theorem 3.3.** *A convex subset $P$ of the real line $\mathbb{R}$ has the order-FPP if and only if $P$ is bounded and has the crop-FPP.*

*Proof.* It follows from the definition that order-FPP implies crop-FPP. If $P$ is not bounded, say not bounded above, then $P$ contains an interval of the form $[a,\ \infty)$. Then the map $f : P \to P$ defined by

$$f(x) = \begin{cases} a & \text{if } x < a; \\ x+1 & \text{if } x \geq a. \end{cases}$$

is a fixed point free order preserving map. Conversely, suppose that $P$ is bounded and has the crop-FPP. If there is a fixed point free order preserving map on $P$, then by Tarski's theorem [10], $P$ is not order-complete. Since $P$ is bounded, either $P$ has no maximum or $P$ has no minimum, say no maximum. If $a = \sup P$ in $\mathbb{R}$, then $P \cup \{a\}$ is convex and hence by Lemma 3.1, $P$ does not have the crop-FPP, a contradiction. □

The above result doesn't hold if $P$ is not convex. For example, $P = [0,\ 1) \bigcup (2,\ 3]$ is bounded and has the crop-FPP; but does not have the order-FPP.

### *3.2. crop-FPP for subsets of* $\mathbb{R}^2$

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**Notation:** For any $X \subseteq \mathbb{R}^2$, denote, $int(X)$= interior of $X$, $\overline{X}$ = closure of $X$ and $bd(X)$ = boundary of $X$. For any $a, b \in \mathbb{R}^2$, denote $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$
$[a,\ b] = \{x \in \mathbb{R}^2;\ a \leq x \leq b\}$

In this section, we characterize crop-FPP for bounded convex subsets of $\mathbb{R}^2$. Through out this section, $P$ denotes a bounded convex subset of $\mathbb{R}^2$, $g = (g_1, g_2) = \inf P$ in $\mathbb{R}^2$ and $l = (l_1, l_2) = \sup P$ in $\mathbb{R}^2$.

**Lemma 3.4.** *If $P$ has the crop-FPP, then $bd(P) \bigcap int([g,\ l]) \subseteq P$.*

*Proof.* The conclusion holds for any line segment in $\mathbb{R}^2$. Thus we may assume that $int(P) \neq \emptyset$.

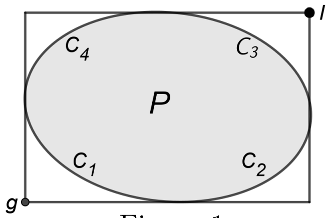

Figure 1

We may split $bd(P) \bigcap int([g,\ l])$ into four components $C_1, C_2, C_3$ and $C_4$ as shown in the Figure 1. (There are cases with fewer components, but the argument is same.) More precisely $C_1$ is the set of all minimal elements of $\overline{P}$ inside $int([g,\ l])$ and $C_3$ is the set of all maximal elements of $\overline{P}$ inside $int([g,\ l])$ in the usual componentwise order. Similarly $C_2$ is the set of all minimal elements of $\overline{P}$ inside $int([g,\ l])$ and $C_4$ is the set of all maximal elements of $\overline{P}$ inside $int([g,\ l])$ in the order $\leq'$ on $\mathbb{R}^2$ defined by $(x_1, x_2) \leq' (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \leq y_2$.

Suppose there is a point $x \in C_1$ that is not in $P$. Choose $y \in P$ close to $x$ with $x \leq y$ so that $[x,\ y] - \{x\} \subseteq P$. The best approximation $f_1$ of $P$ onto the closed rectangle $[x,\ y]$ (See Figure 2) is a crop-retraction with crop-retract $[x,\ y] - \{x\}$. By Lemma 3.1, $[x,\ y] - \{x\}$ does not have the crop-FPP and hence by Theorem 2.7, $P$ does not have the crop-FPP, a contradiction to our assumption. Thus $C_1 \subseteq P$. Similarly $C_3 \subseteq P$.

Suppose there is point $x = (x_1, x_2) \in C_2$ that is not in $P$. Choose a point $y = (y_1, y_2) \in P$ with $y_1 < x_1$ and $x_2 < y_2$ so that $[x \wedge y,\ x \vee y] - \{x\} \subseteq P$. As in the above case, the best approximation $f_2$ of $P$ onto the closed rectangle $[x \wedge y,\ x \vee y]$ (See Figure 3) is a crop-retraction, with crop-retract $[x \wedge y,\ x \vee y] - \{x\}$. By Lemma 3.1, $[x \wedge y,\ x \vee y] - \{x\}$ does not have the crop-FPP and hence by Theorem 2.7, $P$ does not have the crop-FPP, contradicting our assumption. Thus $C_2 \subseteq P$. Similarly $C_4 \subseteq P$.

This proves the Lemma. □

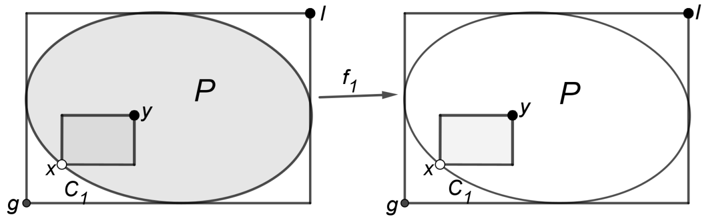


Figure 2

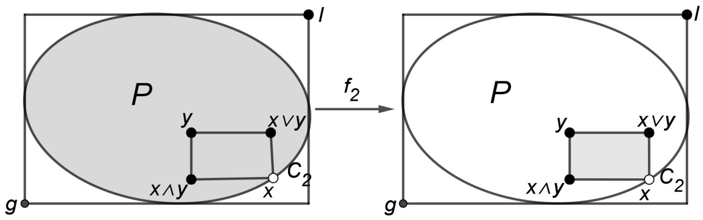


Figure 3

**Lemma 3.5.** *If $P$ has the crop-FPP, then $P$ is doubly chain-complete and each element $p \in \overline{P}$ is comparable to some element of $P$.*

*Proof.* If $P$ is not doubly chain-complete, then $P$ contains a non-empty chain $C$ that does not have a supremum or infimum in $P$, say no supremum. Let $u = (u_1, u_2) = \sup_{\mathbb{R}^2} C$. If $u \in P$, then $u = \sup_P C$, a contradiction. Hence $u \notin P$ so that $u \in bd(P)$. By Lemma 3.4, $u \in bd([g,\ l])$, which consists of four line segments as shown in Figure 4.

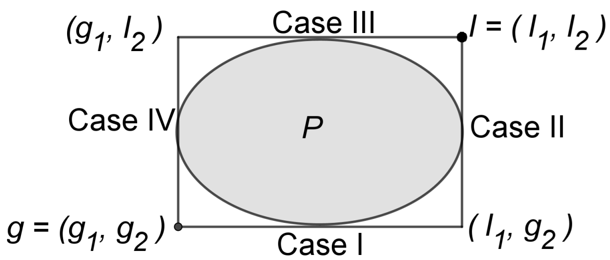


Figure 4

**Case I:** $g_1 \leq u_1 < l_1$ and $u_2 = g_2$
**Case II:** $u_1 = l_1$ and $g_2 \leq u_2 \leq l_2$
**Case III:** $g_1 \leq u_1 \leq l_1$ and $u_2 = l_2$
**Case IV:** $u_1 = g_1$ and $g_2 \leq u_2 < l_2$

Since $u = \sup_P C$, in Case I and Case IV, $P \cup \{u\}$ is convex and hence by Lemma 3.1, $P$ does not have the crop-FPP, a contradiction.

In Case II, denote $S = \{x \in P; x \geq u\}$. If $S \neq \emptyset$, then $v = (v_1, v_2) = \inf_{\mathbb{R}^2} S \notin P$. In this case, $P \cup \{v\}$ is convex and hence by Lemma 3.1, $P$ does not have the crop-FPP, a contradiction. Thus $S = \emptyset$. If $u = (l_1, g_2)$, then $P \cup \{u\}$ is convex and hence by Lemma 3.1, $P$ does not have the crop-FPP, a contradiction. Hence $g_2 < u_2$. Choose a point $p = (p_1, p_2) \in P$ such that $p_1 < u_1$ and $p_2 < u_2$. Let $m$ be the slope of the line $L$ passing through $p$ and $u$. Let $h_1$ be the map that projects all points in $P$ above the line vertically to line $L$ and all other points perpendicularly to $L$ as shown in Figure 5. Then $h_1$ is order preserving and $d(h_1(x), h_1(y)) \leq \sqrt{m^2+1}\, d(x, y)\ \forall\ x, y \in P$. Let $h_2 : L \to L$ be defined by

$$h_2((1-t)p + tu) = \begin{cases} \frac{1}{2\sqrt{m^2+1}}p + (1 - \frac{1}{2\sqrt{m^2+1}})u & \text{if } t \leq 0 \\ (1-t)[\frac{1}{2\sqrt{m^2+1}}p + (1 - \frac{1}{2\sqrt{m^2+1}})u] + tu & \text{if } t > 0 \end{cases}$$

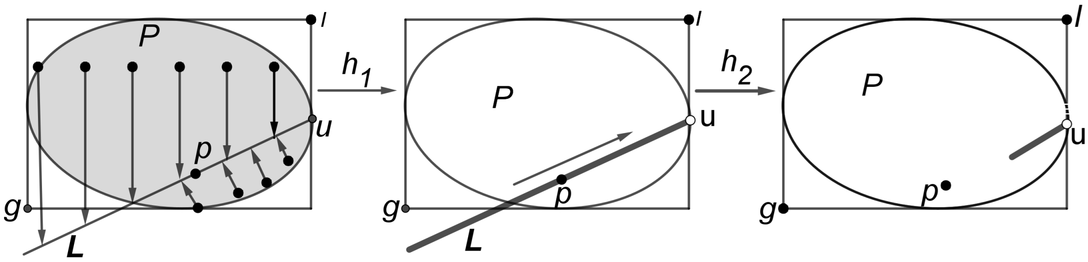


Figure 5

Then $h_2$ is order preserving and $d(h_2(x), h_2(y)) \leq \frac{1}{2\sqrt{m^2+1}} d(x, y) \forall x, y \in L$.

The composite map $h = h_2 \circ h_1$ is an order preserving map on $P$ with contraction constant $\frac{1}{2}$, which does not fix any point of $P$, a contradiction to our assumption. Thus Case II does not arise. By similar arguments, it follows that Case III also does not arise.

This shows that $P$ is doubly chain-complete.

To show the second part, if $p \in \overline{P}$ and there are no points $x \in P$ satisfying $x \leq p$ or $x \geq p$, then by Lemma 3.4, $p \in bd([g,\ l])$. But then $P \cup \{p\}$ is convex and hence by Lemma 3.1, $P$ does not have the crop-FPP, a contradiction. This completes the proof.

□

**Lemma 3.6.** *If $P$ is doubly chain-complete and each $p \in \overline{P}$ is comparable to some element of $P$, then $P$ has the crop-FPP.*

*Proof.* Let $f : P \to P$ be an order preserving contraction. For any $x \in P$, the sequence $\{f^n(x)\}$ is a cauchy sequence and hence it converges to a point $u = (u_1, u_2) \in \mathbb{R}^2$. As $f$ is a contraction, the point $u$ is independent of the choice of $x$.

Suppose $u \notin P$.

**Claim**: There is an element $y \in P$ which is comparable to $f^m(y)$ for some natural number $m$.

If there is an element $x = (x_1, x_2) \in P$ such that $x_1 < u_1$ and $x_2 < u_2$, then as $f^n(x)$ converges to $u$, for $\epsilon = \min\{u_1 - x_1, u_2 - x_2\}$, there exists $m \in \mathbb{N}$ such that $d(f^m(x), u) < \epsilon$ so that $f^m(x) \leq x$. Similarly if there is an element $x = (x_1, x_2) \in P$ such that $u_1 < x_1$ and $u_2 < x_2$, then $x \leq f^m(x)$ for some $m \in \mathbb{N}$

Since $u \notin P$ and $P$ is connected, we are left with two cases, namely $u = (l_1, g_2)$ or $u = (g_1, l_2)$. Suppose $u = (l_1, g_2)$. Since $u \in \overline{P}$, there exists $x \in P$ such that $x \leq u$ or $u \leq x$. If $x \leq u$, then $C = \{x \in P; x \leq u\}$ is a chain and hence it has a supremum in $P$, call it $v$. If $u = \sup_{\mathbb{R}^2} C$, then $u < v$. Otherwise $v < u$. Similarly, when $u \leq x$, there exists $v \in P$ such that $u < v$ or $v < u$. Since $\{f^n(v)\}$ converges to $u$, $v$ is comparable to $f^m(v)$ for $m$ sufficiently large.

Similarly, when $u = (g_1, l_2)$, we can find $v \in P$ such that $v$ is comparable to $f^m(v)$ for some $m \in \mathbb{N}$. This proves the claim.

Since $P$ is doubly chain-complete and $y \sim f^m(y)$, by Theorem 9 in [6], the order preserving map, $f^m : P \to P$ has a fixed point, say $f^m(x_0) = x_0$. The sequence $\{f^n(x_0)\}$ converges to $u$ and its subsequence $\{(f^m)^n(x_0)\}$ converges to $x_0$. Thus $u = x_0 \in P$, a contradiction.

Thus we conclude that $u \in P$ and hence $f(u) = u$. This completes the proof of the lemma. □

From Lemma 3.5 and Lemma 3.6, we get the following main theorem of this section.

**Theorem 3.7.** *A bounded convex subset $P$ of $\mathbb{R}^2$ has the crop-FPP if and only if $P$ is doubly chain-complete and each element $p \in \overline{P}$ is comparable to some element of $P$.*

**Example 3.8.** *Let $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$, the unit closed disk and $I = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq 1, 0 \leq y \leq 1\}$, the unit square.*
*Then both $D$ and $I$ are complete and hence they have the crop-FPP.*

*If any portion of the boundary of $D$ is removed, then the resulting convex set fails to have the crop-FPP as it is not chain-complete. This is not the case with $I$. It depends on the points removed. In fact $\{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < 1\} \cup \{(0, 0), (1, 1)\}$ has the crop-FPP, where as $\{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < 1\} \cup \{(1, 0), (0, 1)\}$ does not have the crop-FPP.*

**Remark 3.9.** *Double chain-completeness alone does not guarantee the crop-FPP. In fact the set $\{(t, 1-t); 0 < t < 1\}$ is doubly chain-complete but does not have the crop-FPP.*

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