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# Some common fixed point theorems for weakly compatible mappings on cone Banach space

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**Abstract**

The primary objective of this paper is to establish the existence of coincidence points and common fixed points for an even number of self-mappings defined on cone Banach spaces under the framework of weak compatibility. By employing an ordered structure and suitable contractive conditions, we present general fixed point results that extend and unify several known theorems in the existing literature. Specifically, Corollaries 3.2, 3.3 and 3.4 address the existence of coincidence and common fixed points for eight, six, and four self-mappings, respectively. Our findings contribute to the ongoing development of fixed point theory in cone metric spaces, offering broad generalizations and encompassing a wide range of previously established results as special cases.

*Keywords:* Coincidence Point, Fixed Point, Compatible Mappings, Cone Banach Space
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## 1. Introduction

The theory of fixed points has emerged as a fundamental area of mathematical analysis due to its extensive applications in various fields such as differential equations, optimization, and dynamic systems. In recent years, the concept of cone metric spaces, as introduced by Huang and Zhang [4], has provided a fruitful generalization of classical metric spaces by replacing the range of the metric with an ordered Banach space.

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This extension enables a more flexible framework to explore fixed point theorems, especially in the absence of continuity or commutativity assumptions.

Following this pioneering work, numerous researchers have investigated the existence and uniqueness of fixed points in cone metric and cone Banach spaces under various contractive conditions. Abbas and Jungck [1] initiated an important line of inquiry by establishing common fixed point results for noncommuting mappings without the assumption of continuity. Subsequent studies have deepened and broadened this framework. For instance, Karapinar [5] and Abdeljawad et al. [2] explored fixed point theorems in cone Banach spaces, contributing significantly to the development of the theory with generalized contraction conditions.

Further advancements were made by Gujetiya et al. [3], who studied compatible mappings, and by Tiwari and Shukla [3], who addressed coincidence and common fixed points within cone Banach space. Sarkar and Tiwary [7] introduced results concerning weakly compatible mappings, while Varghese and Dersanambika [8] offered refinements that underscore the importance of topological structure in determining the existence of fixed points.

This expanding field of research demonstrates significant potential for broad application and further development. Inspired by these advancements, this paper seeks to establish new common fixed point results for $N$ self-mappings under specific conditions within cone Banach spaces, thereby enhancing the theoretical foundation of fixed point theory and its applications.

## 2. Preliminaries

In this section, we outline key definitions and preliminary results that support the development of our main work. These foundational concepts are essential for building the theoretical framework and ensuring a clear understanding of the results presented in the later sections.

**Definition 2.1.** [8] Let $E$ be a real Banach space and let $K \subset E$. The set $K$ is said to be a cone if it satisfies the following conditions:

1. $K$ is nonempty, closed, and $K \neq \{0\}$.
2. For all $x, y \in K$ and for all scalars $a, b \geq 0$, the linear combination $ax + by \in K$.
3. If $x \in K$ and $-x \in K$, then $x = 0$; in other words, $K \cap (-K) = \{0\}$.

Given a cone $K \subset E$, we can define a partial ordering $\leq$ on $E$ by stating that $x \leq y$ if and only if $y - x \in K$. Furthermore, we write $x < y$ when $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of $K$.

Now, let $X$ be a nonempty set and $K \subset E$ as above. A function $d : X \times X \to E$ is called a cone metric if it satisfies the following properties:

1. $d(x, y) \in K$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A pair $(X, d)$, where $d$ is a cone metric, is referred to as a cone metric space.

**Definition 2.2.** [5] Let $E$ be a real Banach space and let $K \subset E$ be a cone with $\operatorname{int}(K) \neq \phi$. A mapping $\|\cdot\| : X \to E$ defined on a real linear space $X$ is called a cone norm if it satisfies the following conditions for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

(i) $\|x\| \geq \theta$ in $E$, and $\|x\| = \theta$ if and only if $x = 0$,
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$,

where "$\geq$" and "$\leq$" denote the partial ordering induced by the cone $K$, and $\theta$ is the zero element of $E$.

The pair $(X, \|\cdot\|)$ is then called a cone normed linear space.

**Definition 2.3.** [5] Let $(X, \|\cdot\|)$ be a cone normed space, where $x \in X$ and $\{x_n\}$ is a sequence in $X$. Then we define the following concepts:

(a) The sequence $\{x_n\}$ is said to converge to $x \in X$ if, for every element $c \in E$ with $0 \ll c$, there exists a natural number $N_1$ such that $\|x_n - x\| \ll c$ for all $n > N_1$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(b) The sequence $\{x_n\}$ is called a Cauchy sequence if, for every $c \in E$ with $0 \ll c$, there exists a natural number $N_1$ such that $\|x_n - x_m\| \ll c$ for all $n, m \geq N_1$.

(c) The cone normed space $(X, \|\cdot\|)$ is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$; that is, every Cauchy sequence has a limit in $X$.

A complete cone normed space is called a cone Banach space.

**Definition 2.4.** [7] Let $(X, \|\cdot\|)$ be a cone normed space. Two self-mappings $A_1$ and $A_2$ on $X$ are said to be compatible if, for every sequence $\{x_n\}$ in $X$ satisfying

$$\lim_{n\to\infty} A_1 x_n = \lim_{n\to\infty} A_2 x_n = x \quad \text{for some } x \in X,$$

it follows that

$$\lim_{n\to\infty} \|A_1 A_2 x_n - A_2 A_1 x_n\| = 0.$$

**Definition 2.5.** [7] Two maps $A_1$ and $A_2$ are called commuting if $A_1 A_2 x = A_2 A_1 x$ for all $x \in X$.

**Definition 2.6.** [8] Let $A_1$ and $A_2$ be two self maps on a set $X$, if $A_1 x = A_2 x$ for some $x$ in $X$ then $x$ is called coincidence point of $A_1$ and $A_2$.

**Definition 2.7.** [7] Let $A_1$ and $A_2$ be self-maps on a cone normed space $(X, \|\cdot\|)$. The mappings $A_1$ and $A_2$ are said to be weakly compatible if they commute at their point of coincidence; that is, if there exists $x \in X$ such that $A_1 x = A_2 x$, then it follows that

$$A_1 A_2 x = A_2 A_1 x.$$

## 3. Main Results

In this section, we present fixed point theorems for an even number of self-mappings in cone Banach spaces. These results extend and generalize the findings of [7] and related works. Our theorems contribute to the broader understanding of fixed point theory by offering new insights and conditions under which common fixed points exist in the framework of cone Banach spaces.

**Theorem 3.1.** *Let $(X, ||.||)$ be a cone Banach space and $d : X \times X \to E$ with $d(x, y) = ||x - y||$. Let $A_1$, $A_2$, ..., $A_N$ be $N$ self mappings on $X$, where $N$ is an even number, satisfying the following conditions:*

(a) $A_N(X) \subseteq A_1 A_2 \ldots A_{\frac{N}{2}-1}(X)$ *and* $A_{N-1}(X) \subseteq A_{\frac{N}{2}} A_{\frac{N}{2}+1} \ldots A_{N-2}(X)$

(b)
$$\begin{aligned} & a||A_{N-1}x - A_N(y)|| + b\Big\{||A_1 A_2 \ldots A_{\frac{N}{2}-1}x - A_{N-1}(x)|| + ||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \ldots A_{N-2}y - A_N(y)||\Big\} \\ & + c\Big\{||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \ldots A_{N-2}(y) - A_{N-1}x|| + ||A_1 A_2 \ldots A_{\frac{N}{2}-1}x - A_N(y)||\Big\} \\ & \quad \leq r||A_1 A_2 \ldots A_{\frac{N}{2}-1}x - A_{\frac{N}{2}} A_{\frac{N}{2}+1} \ldots A_{N-2}y|| \end{aligned} \tag{1}$$
*for all $x, y \in X$, $0 \leq r < a + 2b + 3c$, $a + b + c \neq 0, r \neq a + 2c$.*

(c) $\left(A_{N-1}, A_1 A_2 \ldots A_{\frac{N}{2}-1}\right)$ *and* $\left(A_N, A_{\frac{N}{2}} A_{\frac{N}{2}+1} \ldots A_{N-2}\right)$ *are weakly compatible.*

(d) *If one of $A_{N-1}(X), A_1 A_2 \ldots A_{\frac{N}{2}-1}(X), A_N(X)$ and $A_{\frac{N}{2}} A_{\frac{N}{2}+1} \ldots A_{N-2}(X)$ is a complete subspace of $X$ then*

   (i) $A_{N-1}$ *and* $A_1 A_2 \ldots A_{\frac{N}{2}-1}$ *have a coincidence point and*

(ii) $A_N$ *and* $A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}$ *have a coincidence point in* $X$.

*Then* $A_1, A_2, \dots A_N$ *have a unique common fixed point in* $X$.

*Proof.* Let $x_0 \in X$ be arbitrary then $A_N(x_0) \in X$. Since $A_N(X) \subseteq A_1A_2\dots A_{\frac{N}{2}-1}(X)$ there exists $x_1 \in X$ such that $A_1A_2\dots A_{\frac{N}{2}-1}(x_1) = A_N(x_0)$ and for $x_1$ there exists $x_2 \in X$ such that $A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(x_2) = A_{N-1}(x_1)$ and so on.
Continuing this process we can define a sequence $\{y_n\}$ in $X$ such that

$$\begin{aligned} y_n &= A_Nx_n = A_1A_2\dots A_{\frac{N}{2}-1}(x_{n+1}) \quad \text{and} \\ y_{n+1} &= A_{N-1}x_{n+1} = A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}x_{n+2} \end{aligned}$$

Now we put $x = x_n$ and $y = x_{n+1}$ in (1) we get,
$a||A_{N-1}(x_n) - A_N(x_{n+1})|| + b\Big\{||A_1A_2\dots A_{\frac{N}{2}-1}(x_n) - A_{N-1}(x_n)|| + ||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(x_{n+1}) - A_N(x_{n+1})||\Big\} + c\Big\{||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(x_{n+1}) - A_{N-1}(x_n)|| + ||A_1A_2\dots A_{\frac{N}{2}-1}(x_n) - A_N(x_{n+1})||\Big\}$
$\le r||A_1A_2\dots A_{\frac{N}{2}-1}(x_n) - A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(x_{n+1})||$

$$\begin{aligned} \text{or, } a||y_n - y_{n+1}|| + b\Big\{||y_{n-1} - y_n|| + ||y_n - y_{n+1}||\Big\} \quad +c \quad &\Big\{||y_n - y_n|| + ||y_{n-1} - y_{n+1}||\Big\} \\ &\le r||y_{n-1} - y_n|| \\ \text{or, } a||y_n - y_{n+1}|| + b\Big\{||y_{n-1} - y_n|| + ||y_n - y_{n+1}||\Big\} &\le r||y_{n-1} - y_n|| - c||y_{n-1} - y_{n+1}|| \\ &\le r||y_{n-1} - y_n|| - c\Big\{||y_{n-1} - y_n|| + ||y_n - y_{n+1}||\Big\} \end{aligned}$$

$$\text{or,} \quad ||y_n - y_{n+1}|| \le \frac{r-b-c}{a+b+c}||y_{n-1} - y_n||$$

$$\text{or,} \quad ||y_n - y_{n+1}|| \le k||y_{n-1} - y_n|| \tag{2}$$

where $k = \frac{r-b-c}{a+b+c}$, $k < 1$ as $r < a + 2b + 3c$.
Proceeding as above we will get,

$$||y_n - y_{n+1}|| \le k||y_{n-1} - y_n|| \le k^2||y_{n-2} - y_{n-1}|| \le \dots \le k^n||y_0 - y_1|| \tag{3}$$

where k<1. Now let $m > n$ then

$$\begin{aligned} ||y_m - y_n|| &\le ||y_n - y_{n+1}|| + ||y_{n+1} - y_{n+2}|| + \dots + ||y_{m-1} - y_m|| \\ &\le (k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1})||y_0 - y_1|| \\ &= \frac{k^n(1-k^m)}{1-k}||y_0 - y_1|| \end{aligned}$$

$$\text{Therefore,} \quad ||y_m - y_n|| \le \frac{k^n}{1-k}||y_0 - y_1|| \tag{4}$$

Let $c > 0$. Then, there exists a $\delta > 0$ such that

$$c + N_\delta(0) \subseteq H,$$

where $N_\delta(0) = \{y \in X : \|y\| \le \delta\}$. Since $0 < k < 1$, there exists a positive integer $N'$ such that for all $n \ge N'$,

$$\frac{k^n(1-k^m)}{1-k}\|y_0 - y_1\| \le \delta.$$

This implies

$$\frac{k^n(1-k^m)}{1-k}\|y_0-y_1\| \in N_\delta(0),$$

and hence,

$$-\frac{k^n(1-k^m)}{1-k}\|y_0-y_1\| \in N_\delta(0).$$

Therefore,

$$c-\frac{k^n}{1-k}\|y_0-y_1\| \in c+N_\delta(0) \subseteq H,$$

which implies

$$\frac{k^n}{1-k}\|y_0-y_1\| \le c \quad \text{for all } n \ge N'.$$

So by definition $\{y_n\}$ is a cauchy sequence in $X$. Since $X$ is complete there exists a $z$ in $X$ such that $\lim\limits_{n\to\infty} y_n = z$ and

$$\begin{aligned} \lim_{n\to\infty} A_N x_n = z &= \lim_{n\to\infty} A_1A_2\dots A_{\frac{N}{2}-1}x_{n+1} \\ &= \lim_{n\to\infty} A_{N-1}x_{n+1} = \lim_{n\to\infty} A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}x_{n+1} = z \end{aligned}$$

Now, suppose that $A_1A_2\dots A_{\frac{N}{2}-1}(X)$ is complete. Then there exists a point $p$ in $X$ such that

$$A_1A_2\dots A_{\frac{N}{2}-1}p = z \tag{5}$$

Now put $x=p$ and $y=x_n$ in (1) we get,

$$\begin{aligned} a||A_{N-1}p - A_N(x_n)|| + b\Big\{||A_1A_2\dots A_{\frac{N}{2}-1}p - A_{N-1}p|| + ||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}x_n - A_N(x_n)||\Big\} \\ +c\Big\{||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(x_n) - A_{N-1}p|| + ||A_1A_2\dots A_{\frac{N}{2}-1}p - A_N(x_n)||\Big\} \\ \le r||A_1A_2\dots A_{\frac{N}{2}-1}p - A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}x_n|| \end{aligned}$$

Taking limit as $n\to\infty$ and using (5) in the above inequality we get,

$$a||A_{N-1}p - z|| + b\Big\{||z - A_{N-1}p|| + ||z-z||\Big\} + c\Big\{||z - A_{N-1}p|| + ||z-z||\Big\} \le r||z-z||$$

$$\begin{aligned} &i.e., \quad (a+b+c)||A_{N-1}p - z|| \le 0 \\ &\text{or,} \quad ||A_{N-1}p - z|| = 0 \quad \text{as} \quad (a+b+c) \ne 0 \end{aligned}$$

$$\text{hence,} \quad A_{N-1}p = z. \tag{6}$$

From (5) and (6) we get,
$A_1A_2\dots A_{\frac{N}{2}-1}p = z = A_{N-1}p$
That is $p$ is a coincidence point of $A_1A_2\dots A_{\frac{N}{2}-1}$ and $A_{N-1}$.
As $A_{N-1}(X) \subseteq A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(X)$, $A_{N-1}p = z$ implies $z \in A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(X)$. Let $u$ in $X$ then

$$A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}u = z \tag{7}$$

Now put $x = x_{n+1}$ and $y=u$ in (1) we get,

$$\begin{aligned} a||A_{N-1}x_{n+1} - A_N(u)|| + b\Big\{||A_1A_2\dots A_{\frac{N}{2}-1}x_{n+1} - A_{N-1}x_{n+1}|| + ||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}u - A_N(u)||\Big\} \\ +c\Big\{||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}(u) - A_{N-1}x_{n+1}|| + ||A_1A_2\dots A_{\frac{N}{2}-1}x_{n+1} - A_N(u)||\Big\} \\ \le r||A_1A_2\dots A_{\frac{N}{2}-1}x_{n+1} - A_{\frac{N}{2}}A_{\frac{N}{2}+1}\dots A_{N-2}u|| \end{aligned}$$

Taking limit as $n \to \infty$ and using (7) in the above inequality we get,
$a||z - A_N(u)|| + b\Big\{||z - z|| + ||z - A_N(u)||\Big\} + c\Big\{||z - z|| + ||z - A_N(u)||\Big\} \leq r||z - z||$

$$i.e., (a + b + c)||z - A_N(u)|| \leq 0$$
$$\text{or,} \quad ||A_N(u) - z|| = 0 \quad \text{as} \quad (a + b + c) \neq 0$$

$$\text{Therefore,} \quad A_N(u) = z \tag{8}$$

From (7) and (8) we get,
$A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(u) = z = A_N(u)$
That is $u$ is a coincidence point of $A_N$ and $A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}$. Since $\big(A_{N-1}, A_1A_2, \dots, A_{\frac{N}{2}-1}\big)$ and $\big(A_N, A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}\big)$ are weakly compatible in $X$.
so

$$A_{N-1}.(A_1A_2, \dots, A_{\frac{N}{2}-1})p = (A_1A_2, \dots, A_{\frac{N}{2}-1}).A_{N-1}p$$

$$\text{Therefore,} \quad A_{N-1}(z) = A_1A_2, \dots, A_{\frac{N}{2}-1}(z) \tag{9}$$

and

$$A_N.(A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2})u = (A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}).A_N u$$

$$i.e., A_N(z) = A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(z) \tag{10}$$

Now put $x = z$ and $y = x_n$ in (1) we get,

$$a||A_{N-1}z - A_N(x_n)|| + b\Big\{||A_1A_2 \dots A_{\frac{N}{2}-1}z - A_{N-1}z|| + ||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}x_n - A_N(x_n)||\Big\}$$
$$+c\Big\{||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(x_n) - A_{N-1}z|| + ||A_1A_2 \dots A_{\frac{N}{2}-1}z - A_N(x_n)||\Big\}$$
$$\leq r||A_1A_2 \dots A_{\frac{N}{2}-1}z - A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}x_n||$$

Taking limit as $n \to \infty$ and using (9) in the above inequality we get,
$a||A_{N-1}z - z|| + b\Big\{||A_{N-1}z - A_{N-1}z|| + ||z - z||\Big\} + c\Big\{||z - A_{N-1}z|| + ||A_{N-1}z - z||\Big\} \leq r||A_{N-1}z - z||$

$$\implies (a + 2c - r)||A_{N-1}z - z|| \leq 0$$
$$\implies ||A_{N-1}z - z|| = 0 \quad \text{as} \quad (a + 2c - r) \neq 0$$

$$\implies \quad A_{N-1}z = z. \tag{11}$$

so from (9) we get, $A_{N-1}z = A_1A_2 \dots A_{\frac{N}{2}-1}z = z$
Now put $x = x_{n+1}$ and $y = z$ in (1) we get,

$$a||A_{N-1}x_{n+1} - A_N(z)|| + b\Big\{||A_1A_2 \dots A_{\frac{N}{2}-1}x_{n+1} - A_{N-1}z|| + ||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}z - A_N(z)||\Big\}$$
$$+c\Big\{||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(z) - A_{N-1}x_{n+1}|| + ||A_1A_2 \dots A_{\frac{N}{2}-1}x_{n+1} - A_N(z)||\Big\}$$
$$\leq r||A_1A_2 \dots A_{\frac{N}{2}-1}x_{n+1} - A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}z||$$

Taking limit as $n \to \infty$ and using (10) in the above inequality we get,

$$a||z - A_N(z)|| + b\Big\{||z - z|| + ||A_N(z) - A_N(z)||\Big\} + c\Big\{||A_N(z) - z|| + ||z - A_N(z)||\Big\} \le r||z - A_N(z)||$$

$$\implies (a + 2c - r)||A_N(z) - z|| \le 0$$

$$\implies ||A_N(z) - z|| = 0 \quad \text{as} \quad (a + 2c - r) \ne 0$$

$$\text{Hence,} \quad A_N(z) = z. \tag{12}$$

From (10) we get, $A_N(z) = z = A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2} z$.

Now put $x = A_{\frac{N}{2}-1} z$ and $y = z$ in (1) we get,

$$a||A_{N-1}(A_{\frac{N}{2}-1}z) - A_N(z)|| + b\Big\{||A_1A_2 \dots A_{\frac{N}{2}-1}(A_{\frac{N}{2}-1}z) - A_{N-1}(A_{\frac{N}{2}-1}z)|| + ||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(z) - A_N(z)||\Big\}$$

$$+c\Big\{||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(z) - A_{N-1}(A_{\frac{N}{2}-1}z)|| + ||A_1A_2 \dots A_{\frac{N}{2}-1}(A_{\frac{N}{2}-1}z) - A_N(z)||\Big\}$$

$$\le r||A_1A_2 \dots A_{\frac{N}{2}-1}(A_{\frac{N}{2}-1}z) - A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}(z)||$$

$$\implies a||A_{\frac{N}{2}-1}(z) - z|| + b\Big\{||A_{\frac{N}{2}-1}(z) - A_{\frac{N}{2}-1}(z)|| + ||z - z||\Big\} + c\Big\{||z - A_{\frac{N}{2}-1}(z)|| + ||A_{\frac{N}{2}-1}(z) - z||\Big\} \le r||A_{\frac{N}{2}-1}(z) - z||$$

$$\implies (a + 2c - r)||A_{\frac{N}{2}-1}(z) - z|| \le 0$$

$$\implies ||A_{\frac{N}{2}-1}(z) - z|| = 0 \quad as \quad (a + 2c - r) \ne 0$$

$$\text{Therefore,} \quad A_{\frac{N}{2}-1}(z) = z \tag{13}$$

In similar way if we continue this process by putting $x = A_i(z)$, where $i = 1, 2, \dots, \frac{N}{2} - 2$ and $y = z$ in (1) we get

$$A_i(z) = z \quad \text{for all} \quad i = 1, 2, \dots, \frac{N}{2} - 2. \tag{14}$$

Now, put $x = z$ and $y = A_{N-2}(z)$ in (1) we get,

$$a||A_{N-1}(z) - A_N A_{N-2}(z)|| + b\Big\{||A_1A_2 \dots A_{\frac{N}{2}-1}(z) - A_{N-1}(z)|| + ||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}A_{N-2}(z) - A_N A_{N-2}(z)||\Big\}$$

$$+c\Big\{||A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}A_{N-2}(z) - A_{N-1}(z)|| + ||A_1A_2 \dots A_{\frac{N}{2}-1}(z) - A_N A_{N-2}(z)||\Big\}$$

$$\le r||A_1A_2 \dots A_{\frac{N}{2}-1}(z) - A_{\frac{N}{2}} A_{\frac{N}{2}+1} \dots A_{N-2}A_{N-2}(z)||$$

$$\implies a||z - A_{N-2}(z)|| + b\Big\{||z - z|| + ||A_{N-2}(z) -_{N-2}(z)||\Big\} + c\Big\{||A_{N-2}(z) - z|| + ||z - A_{N-2}(z)||\Big\} \le r||z - A_{N-2}(z)||$$

$$\implies (a + 2c - r)||z - A_{N-2}(z)|| \le 0$$

$$\implies ||z - A_{N-2}(z)|| = 0 \quad \text{as} \quad (a + 2c - r) \ne 0$$

$$\implies A_{N-2}(z) = z \tag{15}$$

In similar way if we continue this process by putting $x = z$ and $y = A_j(z)$, $j = \frac{N}{2}, \frac{N}{2} + 1, \dots, N - 3$ in (1) we obtain,

$$A_j(z) = z \quad \text{for all} \quad j = \frac{N}{2}, \frac{N}{2} + 1, \dots, N - 3. \tag{16}$$

Hence from equations (11)-(16) we conclude that $z$ is a common fixed point of $A_1, A_2, \ldots, A_N$. Now we prove that $z$ is a unique common fixed point $A_1, A_2, \ldots, A_N$. If possible let there exists another fixed point $w(\neq z)$ such that $A_1w = A_2w = \cdots = A_Nw = w$. Putting $x = z$ and $y = w$ in equation (1) we get,

$$a||A_{N-1}z - A_N(w)|| + b\Big\{||A_1A_2\ldots A_{\frac{N}{2}-1}z - A_{N-1}(z)|| + ||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\ldots A_{N-2}w - A_N(w)||\Big\}$$
$$+c\Big\{||A_{\frac{N}{2}}A_{\frac{N}{2}+1}\ldots A_{N-2}(w) - A_{N-1}z|| + ||A_1A_2\ldots A_{\frac{N}{2}-1}z - A_N(w)||\Big\} \leq r||A_1A_2\ldots A_{\frac{N}{2}-1}z - A_{\frac{N}{2}}A_{\frac{N}{2}+1}\ldots A_{N-2}w||$$
$$\implies a||z-w|| + b\Big\{||z-z|| + ||w-w||\Big\} + c\Big\{||w-z|| + ||z-w||\Big\} \leq r||z-w||$$
$$\implies (a+2c-r)||z-w|| \leq 0$$
$$\implies ||z-w|| = 0 \quad as \quad (a+2c-r) \neq 0$$

Therefore, $z = w$

So the fixed point is unique. Hence $z$ is a unique common fixed point of $A_1, A_2, \ldots, A_N$.

□

**Corollary 3.2.** [7] *Let $(X, ||.||)$ be a Cone Banach Space and $d : X \times X \to E$ with $d(x, y) = ||x - y||$. Let $A$, $B$, $C$, $D$, $K$, $M$, $P$ and $V$ be eight self mappings on $X$ that satisfy the conditions:*

(a) $V(X) \subseteq ABC(X)$ *and* $P(X) \subseteq DKM(X)$.

(b) $$a||Px - Vy|| + b\Big\{||ABCx - Px|| + ||DKMy - Vy||\Big\} + c\Big\{||DKMy - Px|| + ||ABCx - Vy||\Big\}$$
$$\leq r||ABCx - DKMy||; \tag{17}$$

*for all* $x, y \in X$, $0 \leq r < a + 2b + 3c$, $a + b + c \neq 0$, $r \neq a + 2c$.

(c) $(P, ABC)$ *and* $(V, DKM)$ *are weakly compatible.*

(d) *If one of* $P(X), ABC(X), V(X), DKM(X)$ *is a complete subspace of* $X$ *then,*
   (i) $P$ *and* $ABC$ *have a coincidence point and*
   (ii) $V$ *and* $DKM$ *have a coincidence point in* $X$.

*Then* $A$, $B$, $C$, $D$, $K$, $M$, $P$ *and* $V$ *have a unique common fixed point in* $X$.

*Proof.* Putting,

$$A_N = V, \quad A_1 = A, \quad A_2 = B, \quad A_3 = C$$
$$\text{and} \quad A_{N-1} = P, \quad A_{\frac{N}{2}} = D, \quad A_{\frac{N}{2}+1} = K, \quad A_{\frac{N}{2}+2} = M.$$

and $A_4 = A_5 = \cdots = A_{\frac{N}{2}-1} = A_{\frac{N}{2}+3} = \cdots = A_{N-2} = I$(Identity mapping) in our main theorem 3.1 we get result.

□

**Corollary 3.3.** [7] *Let $(X, ||.||)$ be a Cone Banach Space and $d : X \times X \to E$ with $d(x, y) = ||x - y||$. Let $A$, $B$, $D$, $K$, $P$ and $V$ be six self mappings on $X$ that satisfy the conditions:*

(a) $V(X) \subseteq AB(X)$ *and* $P(X) \subseteq DK(X)$.

(b) $$a||Px - Vy|| + b\Big\{||ABx - Px|| + ||DKy - Vy||\Big\} + c\Big\{||DKy - Px|| + ||ABx - Vy||\Big\}$$
$$\leq r||ABx - DKy||; \tag{18}$$

*for all* $x, y \in X$, $0 \leq r < a + 2b + 3c$, $a + b + c \neq 0$, $r \neq a + 2c$.

(c) $(P, AB)$ *and* $(V, DK)$ *are weakly compatible.*

(d) *If one of* $P(X), AB(X), V(X), DK(X)$ *is a complete subspace of* $X$ *then,*
   (i) $P$ *and* $AB$ *have a coincidence point and*

(ii) *$V$ and $DK$ have a coincidence point in $X$.*

*Then $A$, $B$, $D$, $K$, $P$ and $V$ have a unique common fixed point in $X$.*

*Proof.* The proof of the theorem 3.3 is similar as the proof of the theorem 3.2. □

**Corollary 3.4.** *Let $(X, ||.||)$ be a Cone Banach Space and $d : X \times X \to E$ with $d(x, y) = ||x - y||$. Let $A$, $D$, $P$ and $V$ be Four self mappings on $X$ that satisfy the conditions:*

(a) *$V(X) \subseteq A(X)$ and $P(X) \subseteq D(X)$.*

(b)
$$a||Px - Vy|| + b\Big\{||Ax - Px|| + ||Dy - Vy||\Big\} + c\Big\{||Dy - Px|| + ||Ax - Vy||\Big\} \leq r||Ax - Dy||; \tag{19}$$

*for all $x, y \in X$, $0 \leq r < a + 2b + 3c$, $a + b + c \neq 0$, $r \neq a + 2c$.*

(c) *$(P, A)$ and $(V, D)$ are weakly compatible.*

(d) *If one of $P(X), A(X), V(X), D(X)$ is a complete subspace of $X$ then,*

(i) *$P$ and $A$ have a coincidence point and*

(ii) *$V$ and $D$ have a coincidence point in $X$.*

*Then $A$, $D$, $P$ and $V$ have a unique common fixed point in $X$.*

*Proof.* Putting,

$$\begin{aligned} A_N = V, &\quad A_1 = A \\ \text{and} \quad A_{N-1} = P, &\quad A_{\frac{N}{2}} = D \end{aligned}$$

and $A_2 = A_3 = \cdots = A_{\frac{N}{2}-1} = A_{\frac{N}{2}+1} = A_{N-2} = I$(Identity mapping) in our main theorem 3.1 we get the result. □

## 4. Conclusions:

In this paper, we have established new results concerning coincidence points and common fixed points through the framework of weakly compatible mappings in Cone Banach Spaces. The main theorem presented here serves as a significant generalization of various existing results in the current literature, thus broadening the scope and applicability of fixed point theory in cone metric spaces. Furthermore, we have demonstrated the existence of coincidence points and common fixed points for eight mappings in Corollary 3.2, six mappings in Corollary 3.3, and four mappings in Corollary 3.4. These findings contribute to the deeper understanding of the structure of fixed points in more complex settings and offer a foundation for future research in nonlinear analysis and its applications.

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## Authors Contributions

All authors contributed equally to the conceptualization, formulation, and design of the research problem.

## Conflicts of Interest

The authors declare no conflict of interest.

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